Lecture 4: Spectral sparsification in dynamic streams

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Algorithms for massive graphs

Massive networks ubiquitous in data processing

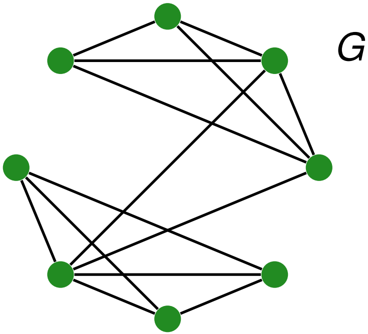
- $\geq$ 100 billion edges
- graph does not fit into memory of single computer
- with metadata, does not fit on a single hard drive

Social distance between nodes, community detection,...

Compress the network while preserving useful properties?

Sparsification

- Let $G = (V, E)$ be an undirected graph, where $|V| = n, |E| = m$.
- Find a smaller subgraph G' of G that approximates G



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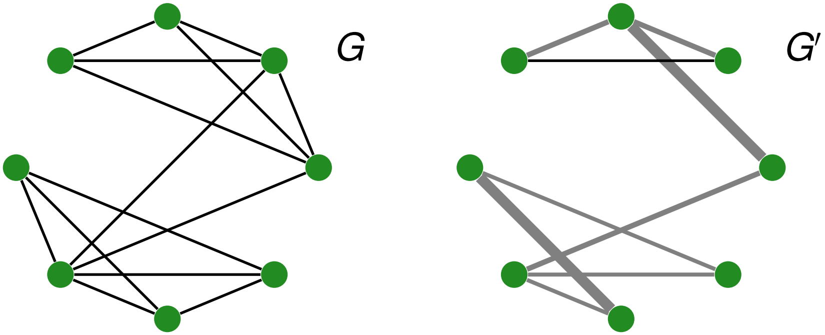

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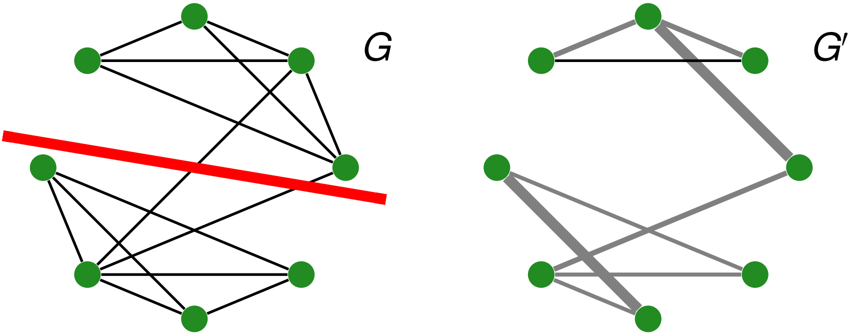

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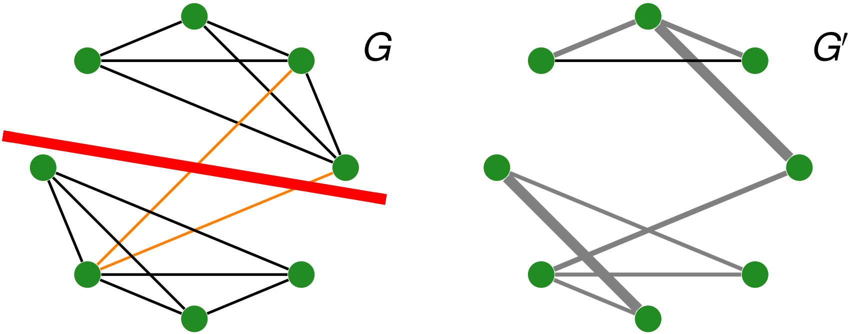

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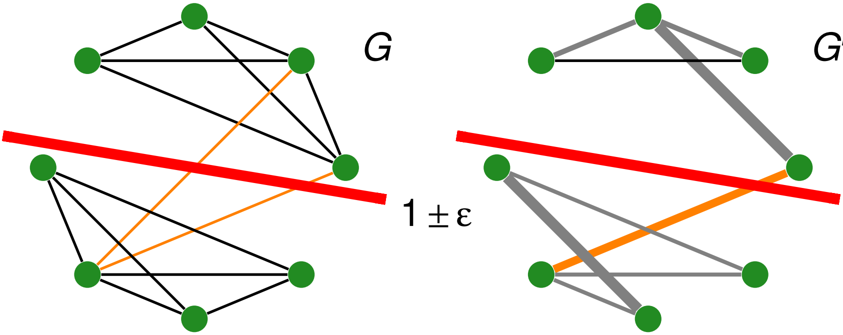
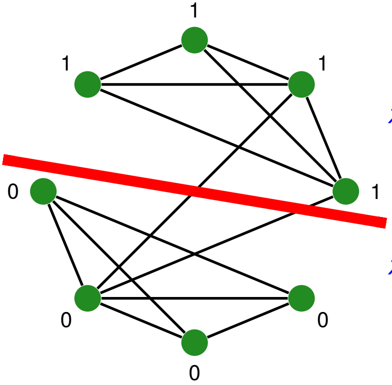


1. Spectral sparsification
2. Streaming model of computation
3. Dynamic streaming and linear sketching
4. Spectral sparsification via linear sketches

1. **Spectral sparsification**
2. Streaming model of computation
3. Dynamic streaming and linear sketching
4. Spectral sparsification via linear sketches

Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2$$



$x_u = 1$

$x_u = 0$

Sparsification

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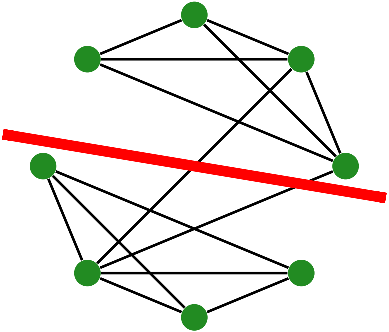
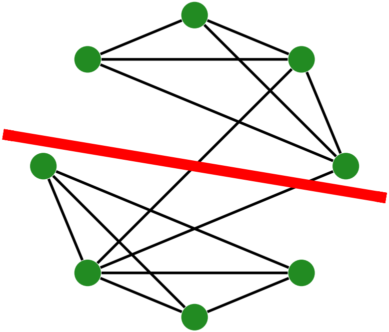


Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = ||z||^2$$



$$z = \begin{bmatrix} 0 \\ x_1 - x_2 \\ 0 \\ x_4 - x_3 \\ 0 \\ x_4 - x_5 \\ \vdots \end{bmatrix}$$

Sparsification

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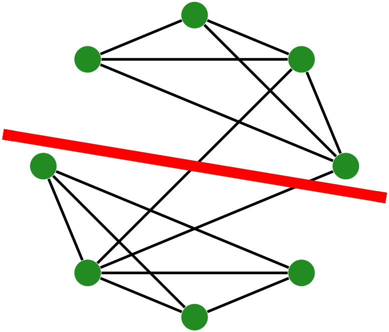
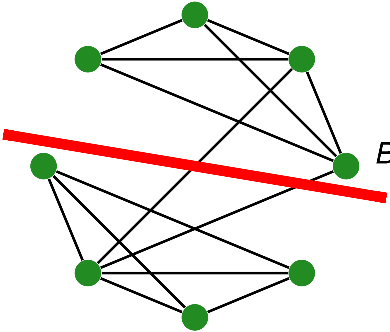


$$z = \begin{bmatrix} 0 \\ x_1 - x_2 \\ 0 \\ x_4 - x_3 \\ 0 \\ x_4 - x_5 \\ \vdots \end{bmatrix} = Bx$$

Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = ||Bx||^2$$

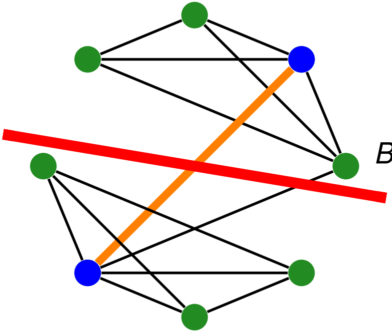


$$Bx = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\binom{n}{2} \times n$

Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = ||Bx||^2$$

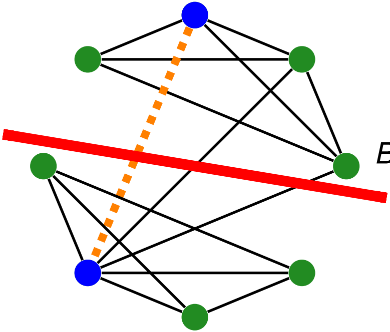


$$Bx = \begin{bmatrix} \mathbf{1} & \mathbf{-1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\binom{n}{2} \times n$

Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = \color{red}{||Bx||^2}$$



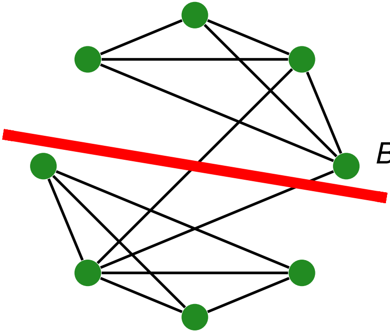
$$Bx = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$\color{red}{\binom{n}{2} \times n}$

Sparsification

$$\text{value of cut} = \sum_{e=(u,v)\in E} (x_u - x_v)^2 = ||Bx||^2 = x^T B^T B x$$

$L = B^T B$ is the Laplacian of G



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$\binom{n}{2} \times n$

Definition (Karger'94, Cut sparsifiers)

G' is an ε-cut sparsifier of G if

$$(1-\varepsilon)x^T Lx \leq x^T L' x \leq (1+\varepsilon)x^T Lx$$

for all $x \in \{0,1\}^V$ (all cuts).

Theorem (Karger'94, Benczur-Karger'96)

For any G there exists an ε-cut sparsifier G' with $O(\frac{1}{\varepsilon^2} n \log n)$ edges, and it can be constructed in $\widetilde{O}(m)$ time.

Definition (Spielman-Teng'04, Spectral sparsifiers)

G' is an ε-spectral sparsifier of G if

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for all ~~$x \in \{0,1\}^V$ (all cuts).~~ all $x \in \mathbb{R}^V$.

Equivalently, $(1-\varepsilon)L \prec L' \prec (1+\varepsilon)L$

Theorem (Spielman-Teng'04, Spielman-Srivastava'09)

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Karger'94, Benczur-Karger'96, Fung-Hariharan-Harvey-Panigrahi'11

Spielman-Teng'04, Spielman-Srivastava'08, Batson-Spielman-Srivastava'09,

Kolla-Makarychev-Saberi-Teng'10, Koutis-Levin-Peng'12, Kapralov-Panigrahy'12

Implications for numerical linear algebra, combinatorial optimization etc

Constructing spectral sparsifiers

Theorem (Spielman-Srivastava'09)

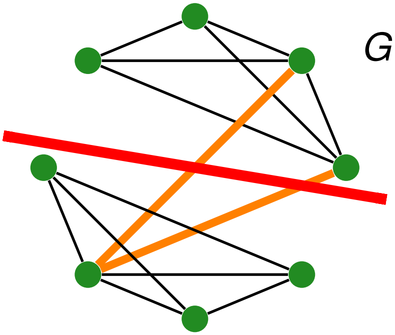
Let $G = (V, E)$ be an undirected graph. Let G' be obtained by including every edge $e \in E$ independently with probability proportional to its effective resistance:

$$p_e \geq \min\{1, \frac{C \log n}{\varepsilon^2} R_e\}.$$

Assigning weight $1/p_e$ if sampled. Then $(1-\varepsilon)L \prec L' \prec (1+\varepsilon)L$ whp.

Sample edges according to a measure of importance, assign weights to make estimate unbiased

Sparsification



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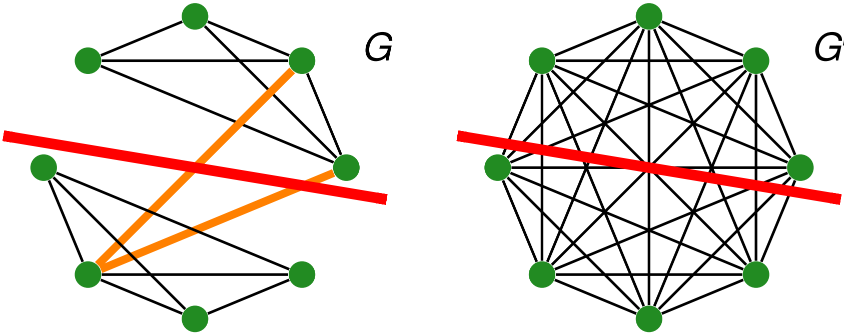


1. Spectral sparsification

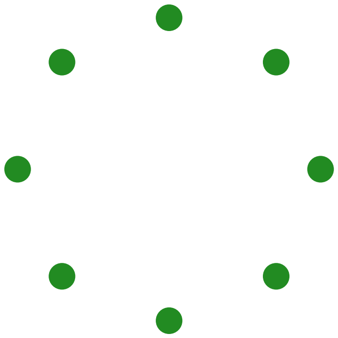
2. **Streaming model of computation**

3. Dynamic streaming and linear sketching

4. Spectral sparsification via linear sketches

Streaming model

- streaming model: edges of G arrive in an arbitrary order in a stream;
- algorithm can only use $\widetilde{O}(n)$ space
- several passes over the stream



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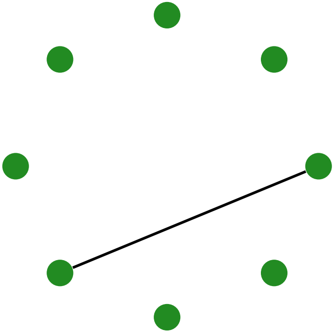

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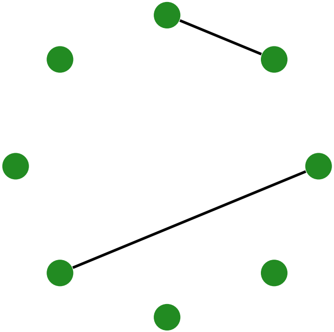

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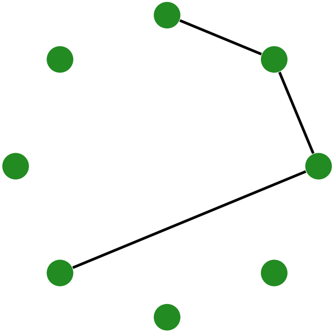

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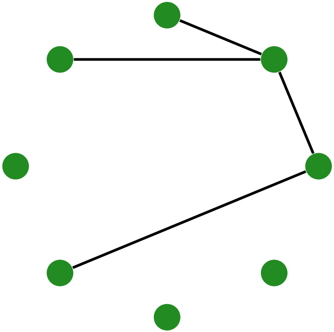

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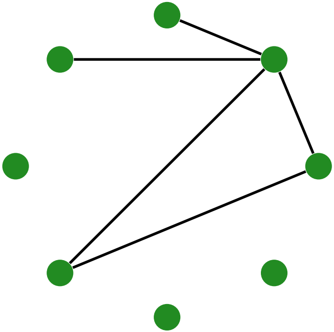

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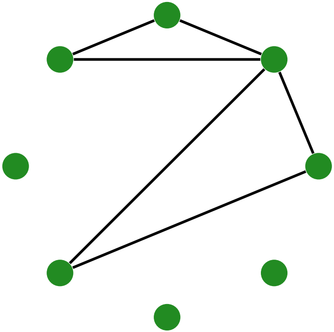

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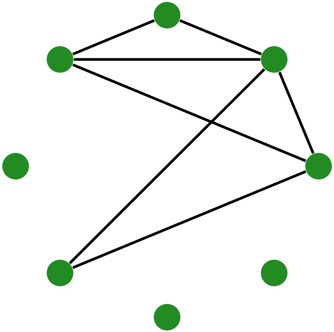

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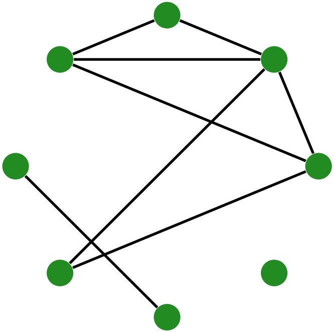

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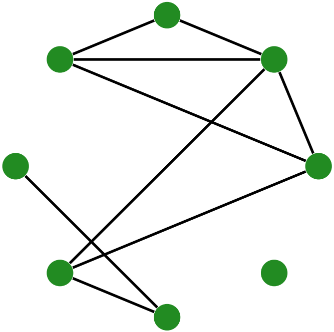

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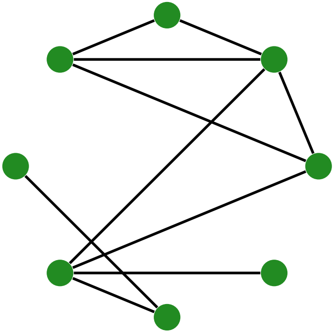

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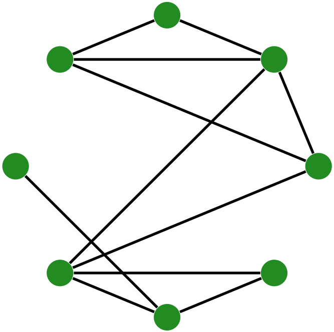

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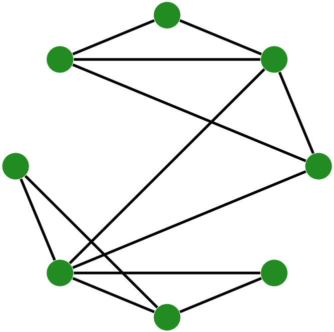

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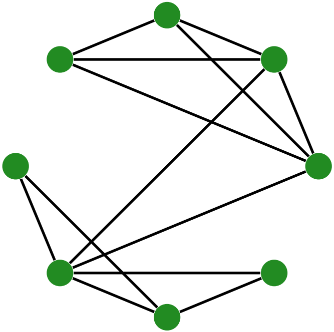

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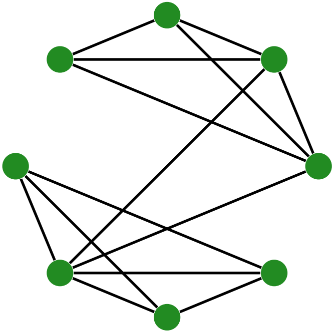

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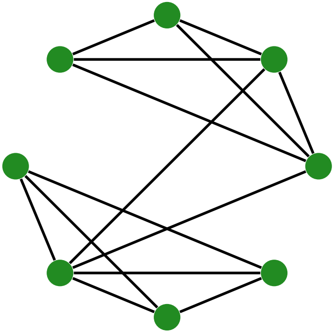
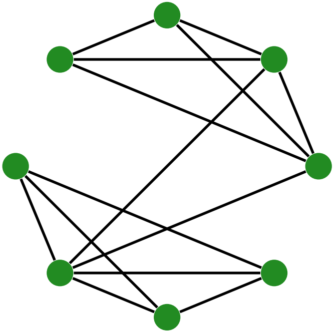


Streaming model

- streaming model: edges of G arrive in an arbitrary order in a stream;
- algorithm can only use $\widetilde{O}(n)$ space
- several passes over the stream **(ideally one pass)**



Insertion-only stream

These algorithms are streamable: just keep resparsifiying the graph as edges come in.

Ahn-Guha'09: $O(\frac{1}{\varepsilon^2} n \log^2 n)$ space for cut sparsifiers

Kelner-Levin'11: $O(\frac{1}{\varepsilon^2} n \log n)$ space for spectral sparsifiers

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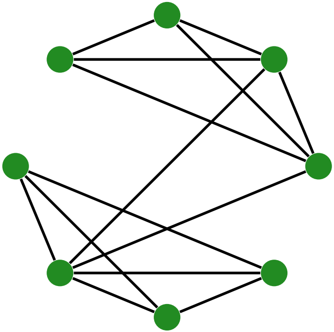
Many modern networks evolve over time, edges both inserted and deleted

Construct sparsifiers in dynamic streams in small space?

1. Spectral sparsification
2. Streaming model of computation
3. **Dynamic streaming and linear sketching**
4. Spectral sparsification via linear sketches (main result)

What if we have deletions?



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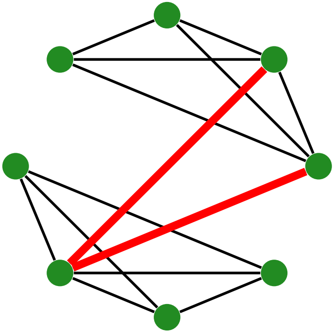

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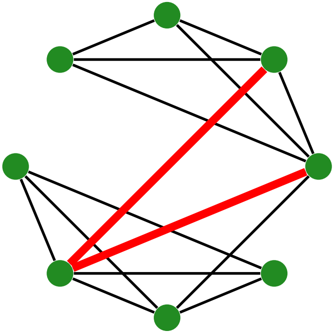

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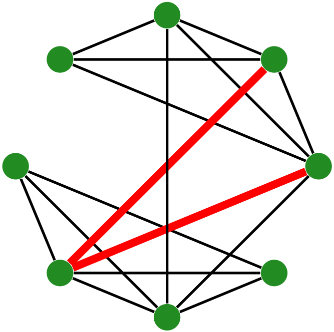

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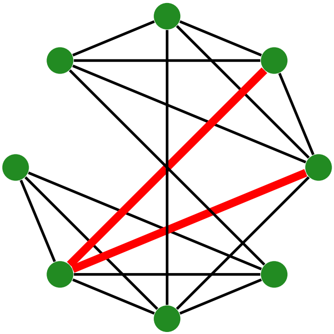

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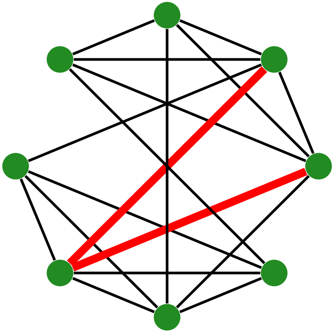

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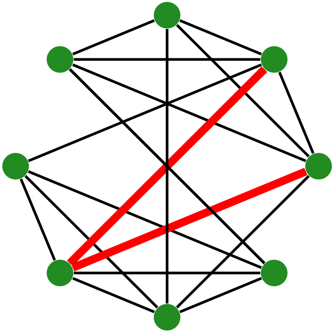

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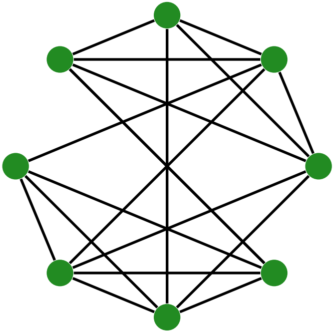

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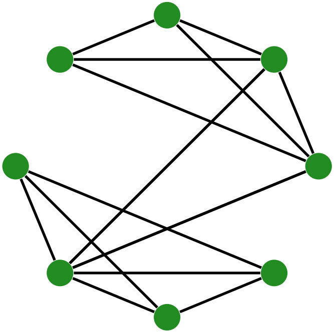

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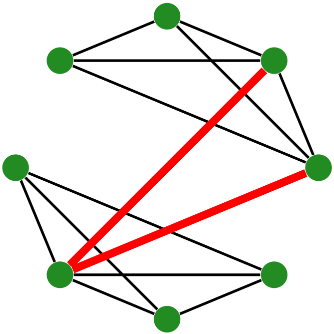

Very different algorithms are needed...

Linear sketching

Classical data stream application: approximating frequency moments.

0	0	0	0	0	0	0	0	0	0	0

Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

Linear sketching

Classical data stream application: approximating frequency moments.

0	0	**1**	0	0	0	0	0	0	0	0

Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

Linear sketching

Classical data stream application: approximating frequency moments.

0	0	1	0	**1**	0	0	0	0	0	0

Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

Linear sketching

Classical data stream application: approximating frequency moments.

0	0	1	0	**2**	0	0	0	0	0	0

Goal: approximate $||x||_2^2 = \sum_i x_i^2$ using $\ll n$ space

Linear sketching

Classical data stream application: approximating frequency moments.

0	0	1	0	**2**	0	0	0	0	0	0

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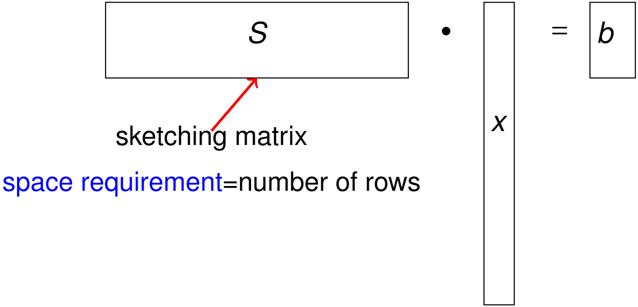
$(1 \pm \varepsilon)$-approximation with $O(\frac{1}{\varepsilon^2} \log n)$ space

Linear sketching

Take (randomized) linear measurements of the input



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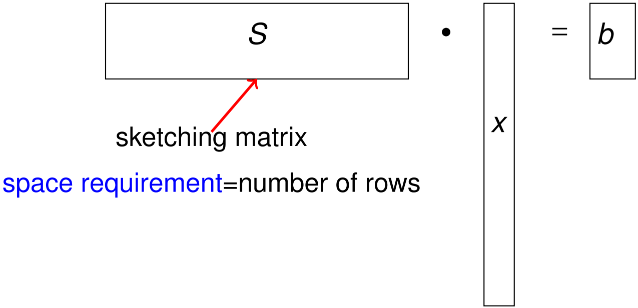

Can get $(1 \pm \varepsilon)$-approximation to $||x||^2$ with $\frac{1}{\varepsilon^2}\text{poly}(\log n)$ rows

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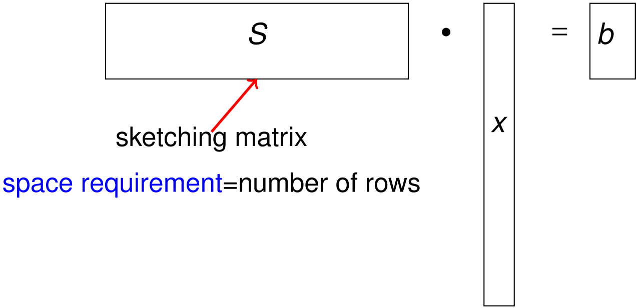

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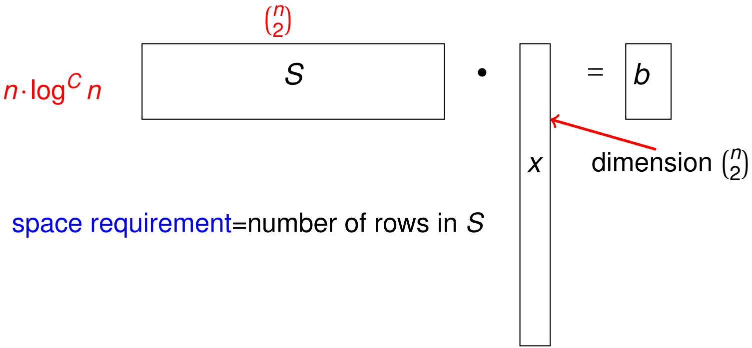
Easy to maintain linear sketches in the (dynamic) streaming model

Graph sketching

Represent adjacency matrix of input graph G as a vector of dimension $\binom{n}{2}$, sketch the vector.

Ahn-Guha-McGregor'SODA12 – connectivity in $n \cdot poly(\log n)$ space.



space requirement=number of rows in S

Graph sketching

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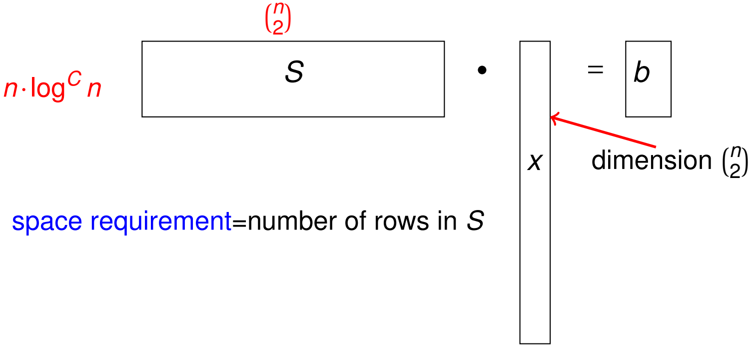

Sketch the adjacency matrix, then reconstruct edges of a sparsifier from the sketch?

	Cut sparsifiers:	Spectral sparsifiers:
Streaming	Ahn-Guha'09	Kelner-Levin'11
Dynamic streaming	Ahn-Guha-McGregor'12 Goel-Kapralov-Post'12 $O(\frac{1}{\varepsilon^2} n\text{poly}(\log n))$ space	Ahn-Guha-McGregor'14 Kapralov-Woodruff'14 $\widetilde{O}(\frac{1}{\varepsilon^2} n^{5/3})$ space $O(\text{poly}(\frac{1}{\varepsilon}) n^{1+o(1)})$ space, two passes

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Theorem (K.-Lee-Musco-Musco-Sidford'14)

There exists a single-pass streaming algorithm that constructs a spectral sparsifier of a graph given as a dynamic stream of edges using $\widetilde{O}(\frac{1}{\epsilon^2} n \mathit{poly}(\log n))$ space and $\mathit{poly}(n)$ runtime.

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Essentially optimal space complexity, oblivious compression scheme

1. Spectral sparsification
2. Streaming model of computation
3. Dynamic streaming and linear sketching
4. **Spectral sparsification via linear sketches**

Constructing spectral sparsifiers

Theorem (Spielman-Srivastava'09)

Let $G = (V, E)$ be an undirected graph. Let G' be obtained by including every edge $e \in E$ independently with probability proportional to its effective resistance:

$$p_e \geq \min\{1, \frac{C \log n}{\varepsilon^2} R_e\}.$$

Assign weight $1/p_e$ if sampled. Then $(1-\varepsilon)G \prec G' \prec (1+\varepsilon)G$ whp.

Sample edges according to a measure of importance, assign weights to make estimate unbiased

Note: edges e with resistance $R_e = \Omega(1/\log n)$ included with probability 1

Constructing spectral sparsifiers offline

Maintain: sketch $S \cdot B$ of the incidence matrix B

Step 1. Compute sampling probabilities p_e for each $e \in E$

Step 2. Sample edges independently with probability p_e, give weight $1/p_e$.

Constructing spectral sparsifiers offline

Maintain: sketch $S \cdot B$ of the incidence matrix B

Step 1. Compute sampling probabilities p_e for each $e \in E$

[Q] How? We do not know which edges are present in the graph...

Step 2. Sample edges independently with probability p_e, give weight $1/p_e$.

[Q] Sample from the sketch?

Refining a sparsifier

Goal: design a sketch S that allows sampling edges of G according to effective resistance given

- $S \cdot B$ (sketch of edge incidence matrix)

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Goal: design a sketch S that allows sampling edges of G according to effective resistance given

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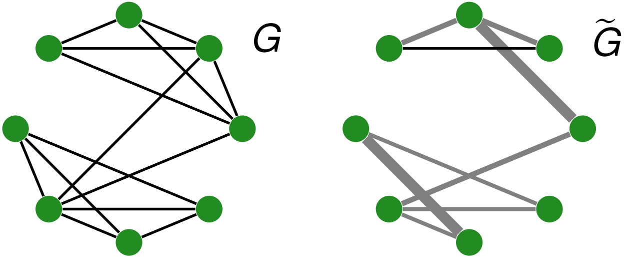
- crude constant factor spectral sparsifier $\widetilde{G}$

$$\frac{1}{C} \cdot L \prec \widetilde{L} \prec L$$



Construct a $1+\varepsilon$ sparsifier G' of G

Refining a sparsifier

Goal: design a sketch S that allows recovery of high resistance ($\geq 1/\log n$) edges of G given

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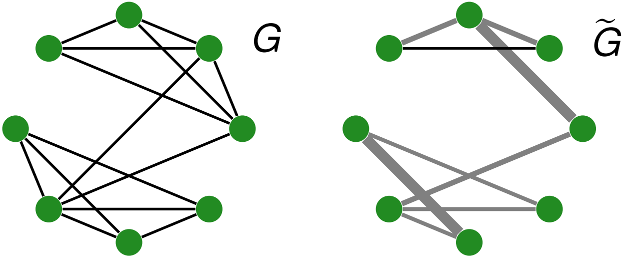
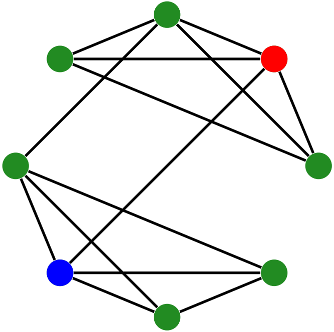


Construct a $1+\varepsilon$ sparsifier G' of G

Effective resistance

$$R_{uv} = b_{uv}^T L^+ b_{uv}$$

Note: defined for any pair (u, v).



Inject current at u, take out at v.

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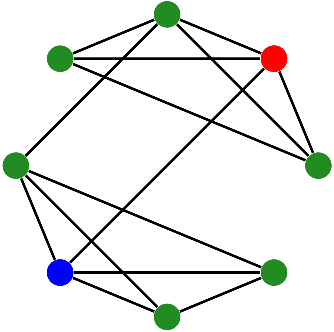

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$\phi = L^+ b_{uv}$=vertex potentials

$$b_{uv} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

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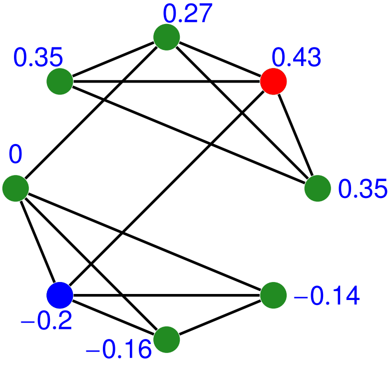

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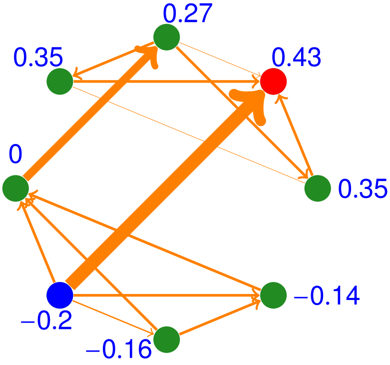

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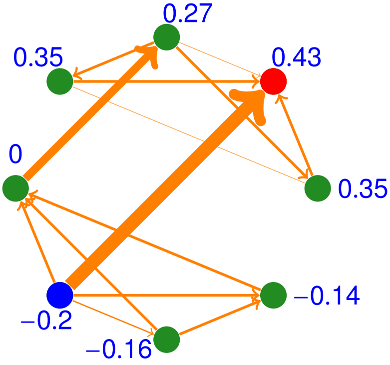

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$f_{xy} = \phi_y - \phi_x = b_{xy}^T L^+ b_{uv}$

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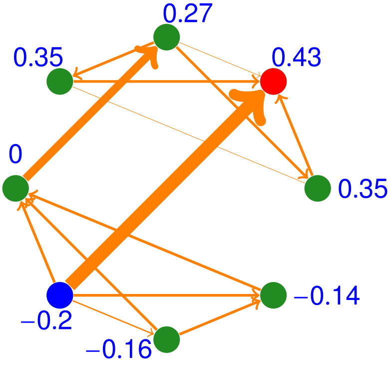

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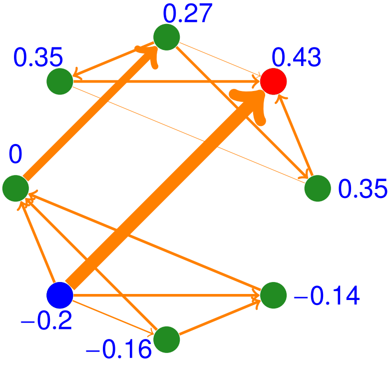

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$$R_e = \frac{f_e^2}{||f||^2}.$$

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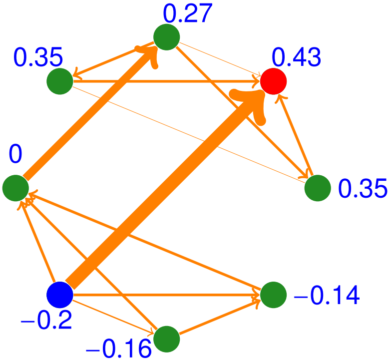

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$$R_e = \frac{f_e^2}{||f||^2}.$$

R_{uv}= fraction of $||f||_2^2$ contributed by $e = (u, v)$

Given:

- a sketch $S \cdot B$ of G
- crude sparsifier $\widetilde{G}$
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Need:

- is (u, v) an edge in G of resistance $\Omega(1/\log n)$?

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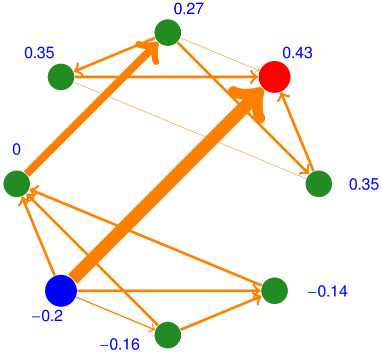

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0	0	0.01	0	**200**	1	0	2	0	0	0

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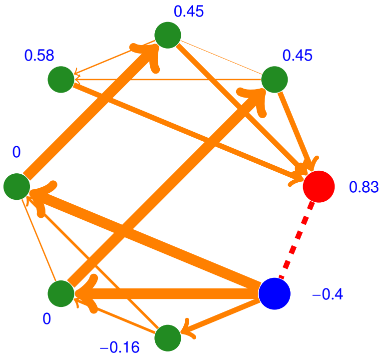

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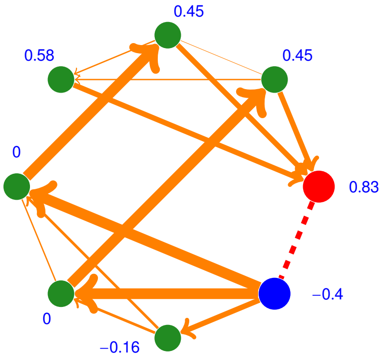

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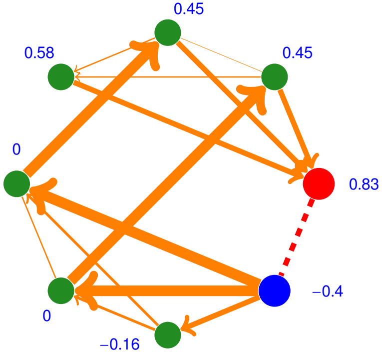

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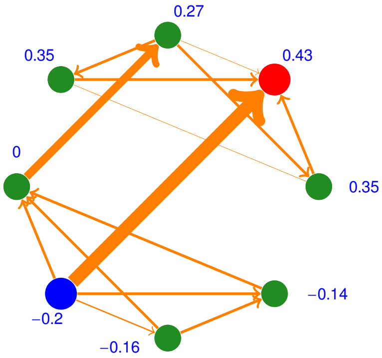

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Linear sketching and sparse recovery

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Let y be a vector of reals. Then $i \in [n]$ is an ℓ_2-heavy hitter if

$$y_i^2 \geq \eta ||y||_2^2.$$

Lemma (ℓ_2-heavy hitters)

For any $\eta > 0$ there exists a (randomized) sketch in dimension $\frac{1}{\eta} poly(\log n)$ from which one reconstruct all η-heavy hitters. The recovery works in time $O(\frac{1}{\eta} \cdot poly(\log n))$.

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Need to recover 'heavy' coordinates of

$$S\Delta := (SB)\phi = S \cdot \begin{bmatrix} \phi_1 - \phi_2 \\ \phi_2 - \phi_3 \\ 0 \\ \phi_4 - \phi_3 \\ \phi_3 - \phi_6 \\ 0 \\ \phi_3 - \phi_1 \\ 0 \\ \vdots \end{bmatrix}$$

A coordinate $e \in \binom{n}{2}$ is heavy if $\Delta_e^2 = \Omega(||\Delta||_2^2/(C \log n))$

This is the ℓ_2 heavy hitters problem!

Problem: we do not know Δ in advance!

Sketching the edge incidence matrix

$$S \cdot B = S \cdot \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & & & & & \vdots \end{bmatrix}$$

Apply ℓ_2-heavy hitters sketch S to every column $b_u, u \in V$ of B

Store the n sketches, $n \cdot \log^C n$ space.

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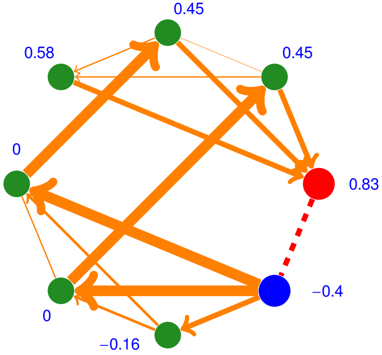
Compute $\phi = \widetilde{\mathbf{L}}^{+} b_{uv}$ – vertex potentials



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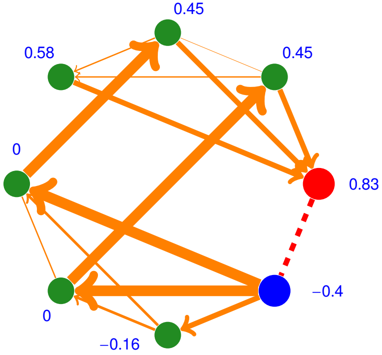

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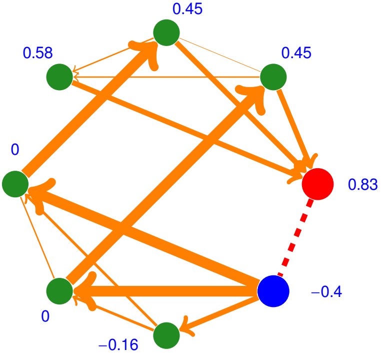

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using heavy-hitters sketch S

0	0	0.01	0	200	1	0	2	0	**0**	0

What about an edge of resistance $r \approx 2^{-j} = o(1)$? it only contributes a $\approx 2^{-j}$ fraction of ℓ_2 mass...

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Sample edges with probability 2^{-j}

If edge (a,b) is in G and is sampled, it contributes $\Omega(\frac{1}{C \log n})$ fraction of mass whp – can recover.

What about an edge of resistance $r \approx 2^{-j} = o(1)$? it only contributes a $\approx 2^{-j}$ fraction of ℓ_2 mass...

Need to recover 'heavy' coordinates of

$$S\Delta := (SB)\phi = S \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ \phi_4 - \phi_3 \\ \phi_3 - \phi_6 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Sample edges with probability 2^{-j}

If edge (a, b) is in G and is sampled, it contributes $\Omega(\frac{1}{C \log n})$ fraction of mass whp – can recover.

Store sketches of subsampled edge incidence matrix:

$$S\Pi_j B, j = 0, \ldots, \log_2 n.$$

Π_j is a diagonal matrix with Bernoulli$(0/1, 2^{-j})$ entries

REFINESPARSIFIER$(G, \widetilde{G}, \varepsilon, c)$

For $e = (a, b) \in \binom{V}{2}$

$\widetilde{R}_e \leftarrow b_e^T \widetilde{L}^+ b_e$ ▷ resistance in $\widetilde{G}$

Round: $\widetilde{R}_e \approx 2^{-j}$ ▷ determine sampling level

$x_e \leftarrow \widetilde{L}^+ b_e$

If TESTEDGE$(S\Pi_j B, x_e, e)$ **then**
add e to sparsifier with weight 2^j

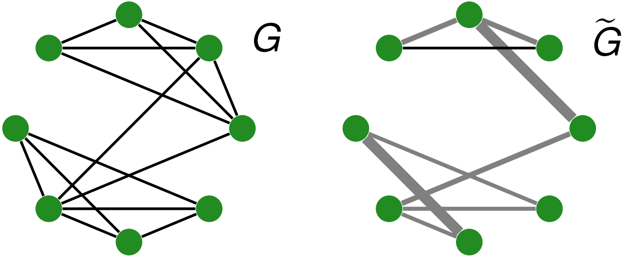
Repeat (C/ε^2) **times, take union**

Refining a sparsifier

Designed a sketch S that allows sampling edges of G according to effective resistance given

- $S \cdot B$ (sketch of edge incidence matrix)
- crude constant factor spectral sparsifier $\widetilde{G}$

$$\frac{1}{C} \cdot L \prec \widetilde{L} \prec L$$

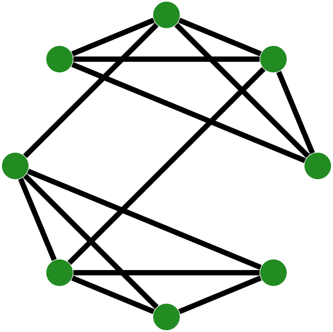


Chain of coarse sparsifiers

Approach of Miller and Peng in Iterative approaches to row sampling

Add a weighted complete graph to G:

$$G(\lambda) = G + \lambda K_n$$



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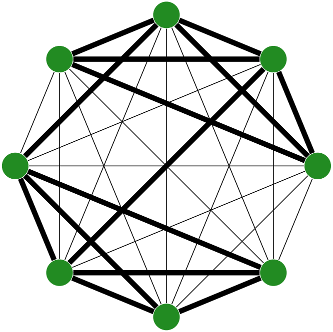

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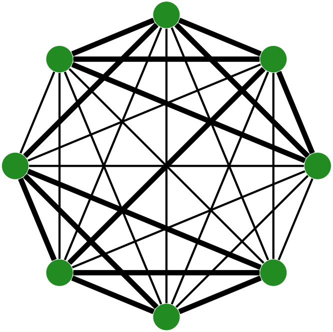

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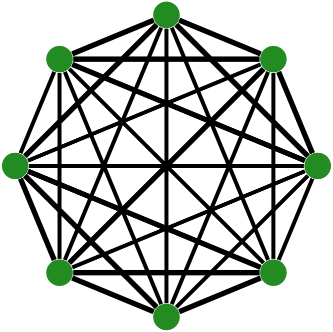

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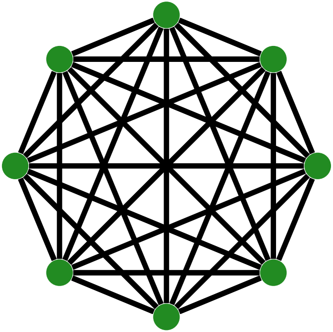

$$L(\lambda) = L + \lambda \cdot nI^*$$

Nonzero eigenvalues of G are between n and $8/n^2$, so

- K_n is a C-spectral approximation to $G(1)$
- $G(1/\text{poly}(n))$ approximates G well spectrally.

Consider powers of 2 from 1 to $1/\text{poly}(n)$.

Two adjacent graphs in the chain are similar:

$$\frac{1}{2}G(\lambda) \prec G(\lambda/2) \prec G(\lambda)$$

This is exactly what we need for RefineSparsifier...

Final algorithm

$$\widetilde{G}_{1/2} \leftarrow \textsc{RefineSparsifier}(S \cdot G(1/2),\ \ K_n, \varepsilon, 3)$$

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Space requirement

- $O(\log n)$ sampling levels, $O(\frac{1}{\varepsilon^2} \log n)$ repetitions
- $O(\log n)$ long chain of coarse sparsifiers
- an ℓ_2-heavy hitters sketch of $O(\text{poly}(\log n))$ size for each node.

Summary

- AMS sketch (approximating $||x||_2^2$)
- Heavy hitters (CountSketch)
- ℓ_0 samplers
- Graph connectivity
- Graph sparsification

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Which other graph problems admit sketching solutions?