18.997 Topics in Combinatorial Optimization	February 12, 2004

Lecture 4

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This lecture covers: the Matching polytope, total dual integrality, and Hilbert bases.

1 The Matching Polytope and Total Dual Integrality

In this section we introduce the matching polytope as the convex hull of incidence vectors of matchings. Next, we give a linear description of the matching polytope, and prove that the linear description is correct by introducing the concept of Total Dual Integrality (TDI).

Given a graph $G = (V, E)$, and a matching M on G, we can identify M with its incidence vector:

$$\chi^M \in \mathbb{R}^{|E|} \quad : \quad \chi^M_e = \begin{cases} 1 & e \in M \\ 0 & \text{otherwise} \end{cases}$$

We define the matching polytope $\mathcal{P} = \mathcal{P}(G)$ to be the convex hull of these incidence vectors:

$$\mathcal{P}(G) = \text{conv}\{\chi^M \,:\, M \text{ a matching of } G.\}$$

We wish to obtain a linear description of $\mathcal{P} \subseteq \mathbb{R}^{|E|}$. We must have $x_e \geq 0$ for $e \in E$. Also, every vertex can have at most one adjacent edge in any matching, and thus

$$x(\delta(v)) \triangleq \sum_{e \in \delta(v)} x_e \leq 1.$$

Thus our first attempt at a linear description is:

$$P^1 = \left\{ \begin{array}{ll} x_e \geq 0 & \forall e \in E \\ x(\delta(v)) \leq 1 & \forall v \in V \end{array} \right\}$$

Consider the triangle, with edges labelled $\{1, 2, 3\}$. In this case, the matching polytope is

$$\mathcal{P} = \text{conv}\{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}.$$

The point $(1/2, 1/2, 1/2)$ is in P^1, i.e., it satisfies the constraints above, however it is not in the convex hull of the matching vectors. This example motivates the following family of constraints.

Observe that for any matching M, any odd cardinality subset U can have at most $(|U| - 1)/2$ edges. Thus we have the additional constraints

$$x(E(U)) = \sum_{e \in E(U)} x_e \leq \frac{|U| - 1}{2}, \qquad U \subseteq V, \ |U| \text{ odd},$$

which we call the odd subgraph constraints. For the triangle, taking U to be the graph itself, we get the constraint $x_1 + x_2 + x_3 \leq 1$. This constraint is violated by the point $(1/2, 1/2, 1/2)$. As a second attempt at a linear description of the matching polytope we have:

$$P^2 = \left\{ \begin{array}{ll} x_e \geq 0 & \forall e \in E \\ x(\delta(v)) \leq 1 & \forall v \in V \\ x(E(U)) \leq \frac{|U|-1}{2} & U \subseteq V, \ |U| \text{ odd} \end{array} \right\}$$

Theorem 1 (Edmonds, 1965) *The linear description P^2 is in fact the Matching polytope, i.e., $\mathcal{P} = P^2$.*

Proof: Edmonds gave an algorithmic proof: For any given weight vector, w, he argued that the optimization over P^2 gives the same maximum weighted matching as the actual solution. The algorithm uses shrinking and expanding of blossoms. It is in Chapter 27 of Schrijver's book. $\square$
Rather than give Edmonds's algorithmic solution, we introduce the concept of Total Dual Integrality (TDI). This, in addition to Total Unimodularity, is further explored in Lecture 5.

Recall the primal and dual standard formulation of an LP.

$$\text{(Primal }(P)) \quad \left\{ \begin{array}{ll} \max : & c^\top x \\ \text{s.t.} : & Ax \leq b \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{ll} \min : & b^\top y \\ \text{s.t.} : & A^\top y = c \\ & y \geq 0 \end{array} \right\} \quad \text{(Dual }(D))$$

We define Total Dual Integrality as follows:

Definition 1 (TDI) *A polyhedron defined by a system of inequalities $\{x : Ax \leq b\}$ (with A and b rational) is Total Dual Integral (TDI) if for any integral cost vector c to the primal problem, if the associated dual*

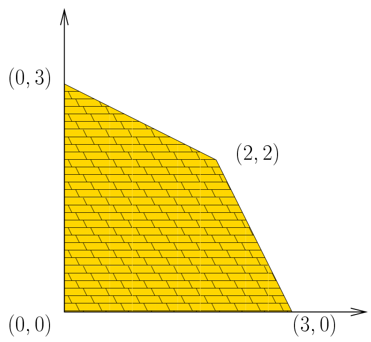
$$\begin{array}{rc} \mathit{min}: & b^\top y \\ \mathit{s.t.}: & A^\top y = c \\ & y \geq 0 \end{array}$$

has a finite optimal value, then it also has an integral solution y^*.

We now state the main theorem about TDI polytopes. We save the proof for the next lecture.

Theorem 2 (Edmonds-Giles, 1978) *If the system $\{Ax \leq b\}$ is TDI, and b is integral, then the polytope $\{Ax \leq b\}$ is integral, i.e., all vertices are integral.*

Total Dual Integrality is a property of the representation of the polyhedron. To illustrate this point, consider the two dimensional polytope



$$\mathcal{P} = \text{conv}\{(0,3), (2,2), (0,0), (3,0)\}$$

This polytope may have many different representations. For example,

$$\mathcal{P} = \left\{ \begin{array}{l} x_1 \geq 0, \;\; x_2 \geq 0 \\ x_1 + 2x_2 \leq 6 \\ 2x_2 + x_1 \leq 6 \end{array} \right\}$$

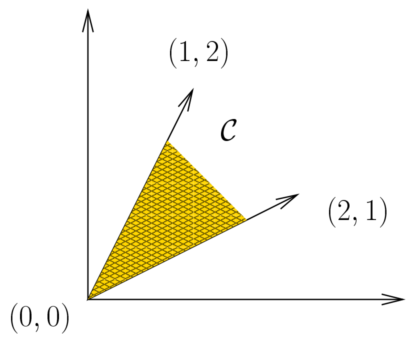
While the polytope is, evidently, integral, the system given is not TDI. For example, consider $c = (1,1)$. The dual to

$$\begin{array}{ll} \min: & c'x \\ \text{s.t.}: & x \in \mathcal{P} \end{array}$$

is the linear program

$$\begin{array}{ll} \min: & 6y_1 + 6y_2 \\ \text{s.t.}: & y_1 + 2y_2 \geq 1 \\ & 2y_1 + y_2 \geq 1 \\ & y_1, y_2 \geq 0 \end{array}$$

The dual optimal value is finite, however there are no integral optimal points. The dual tries to express $(1,1)$ as a positive linear combination of the vectors $(1,2)$ and $(2,1)$. This cannot be done as an integral linear combination. From this example, we see that in order for the dual to have an integral optimal solution, we must be able to express any integral vector in the dual cone $\mathcal{C}$ (as in the figure below) as an integral combination of some set of vectors.



Definition 2 *Given a rational (i.e. of the form $\{x : Ax \leq 0\}$ with A rational) cone $\mathcal{C}$, the minimal set of vectors such that any integral point in $\mathcal{C}$ can be expressed as an integral nonnegative linear combination of vectors in the set, is called a Hilbert Basis for the cone.*

For the cone defined by the vectors $(1,2)$ and $(2,1)$, a Hilbert basis is given by the set of vectors $H = \{(1,2),(2,1),(1,1)\}$. We can get the additional vector $(1,1)$ by adding the redundant constraint $x_1 + x_2 \leq 4$ in the primal. Therefore, the linear system

$$\left\{ \begin{array}{l} x_1, x_2 \geq 0 \\ x_1 + 2x_2 \leq 6 \\ 2x_1 + x_2 \leq 6 \\ x_1 + x_2 \leq 4 \end{array} \right\}$$

is TDI.

Remark: Any rational cone has a finite Hilbert basis. This does not necessarily hold for all irrational cones.

Theorem 3 *For P a rational polyhedron, there exists a TDI system $\{Ax \leq b\}$ such that $P = \{x : Ax \leq b\}$. Furthermore, the polytope P is integral if and only if we can take b to be integral.*

The primal form of the statement says that if P is a rational polyhedron such that $\max\{c^\top x : x \in P\}$ is integral for every integral vector c, then the polyhedron is integral.

Suppose that for every rational supporting hyperplane (i.e., for every α, β such that $\alpha^\top x \leq \beta$ for every $x \in P$, and $\{\alpha^\top x = \beta\} \cap P \neq \emptyset$) there exists an integer point on the hyperplane. Then, the polytope is integral.

This gives us an analog of Farkas's lemma: The system $Ax = b$ has no integral solution if and only if $A^\top y$ integral and $b^\top y$ non-integral has a solution.