18.997 Topics in Combinatorial Optimization	March 11, 2004

Lecture 10

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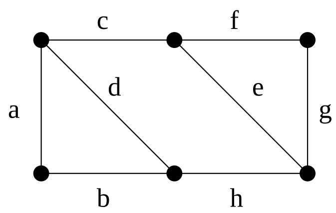
Matroid theory was first formalized in 1935 by Whitney [5] who introduced the notion as an attempt to study the properties of vector spaces in an abstract manner. Since then, matroids have proven to have numerous applications in a wide variety of fields including combinatorics and graph theory.

Today we will briefly survey matroid representation and then discuss some problems in matroid optimization and the corresponding applications. The tools we develop will help us answer the following puzzle:

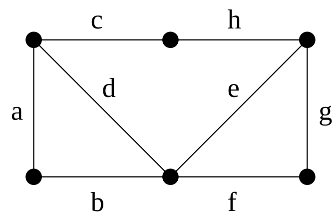
Puzzle: *A game is played on a graph $G(V, E)$ and has two players, George and Ari. Ari's moves consist of "fixing" edges $e \in E$. George's moves consist of deleting any unfixed edge. The game ends when every edge has been either fixed or deleted. Ari wins if the graph at the end of the game is connected (i.e. if the fixed edges form a spanning tree). Otherwise George wins. Supposing George moves first, characterize the graphs in which George has a winning strategy.*

1 Graphic Matroids

Let us begin with some comments regarding graphic matroids that arose during a discussion after class last time. Recall the *graphic matroid* of graph G is $M(G) = (E, \mathcal{I} = \{F \subseteq E : F \text{ is acyclic}\})$. For which graphs G and H does $M(G) = M(H)$? It is easy to see that the matroid representations of two different graphs might be the same. For example, for the graphs G and H in Figure 1(a) and Figure 1(b) respectively, $M(G) = M(H)$.



(a) G



(b) H

Figure 1: Switching operation preserves matroid representation.

We can think of H as being obtained from G by taking a vertex cut of size two and switching the roles of each vertex in one of the subgraphs. In fact, for 2-connected graphs, this operation always preserves the matroid representation.

Theorem 1 *If G and H are 2-connected, then $M(G) = M(H)$ if and only if H can be obtained from G via a sequence of switching operations.*

For higher connectivity, no opertaions exist that lead to the same matroid.

Theorem 2 *If G and H are 3-connected, then $M(G) = M(H)$ if and only if $G = H$.*

In general, graph (vertex) connectivity can be equated to a corresponding notion of matroid connectivity. In particular, it can be shown that a graphic matroid corresponding to a k-connected graph is k-connected and vice versa. However, we will not define matroid connectivity here.

Let us make one final observation concerning graphic matroids. Recall that last time we saw if a graph G is planar (we assumed sufficiently connected, so that it is uniquely embeddable, but this is not necessary), then $M^*(G) = M(G^*)$ where the * operation indicates taking the dual of the corresponding object. It can be shown that planar graphs are unique in this sense.

Theorem 3 (Tutte) *The dual matroid of a graphic matroid $M(G)$ corresponding to graph G is itself a graphic matroid if and only if G is planar.*

2 Matroid Representation

We would like to characterize matroids representable over a finite field. As a first step, note a matroid and its dual are representable over the same fields.

Theorem 4 *If M is representable over F, then so is M^*.*

Proof: Suppose the bases of M have size m. Then, by assumption, M can be represented by an $m \times n$ matrix $A = [I^{m\times m}|B^{m\times(n-m)}]$. The columns of this matrix are indexed by the elements of the ground set. Let Z be a basis of M and rearrange the rows and columns of A such that $Z = X_2 \cup Y_1$ where X_2 and Y_1 are as pictured (Figure 2(a)).

	X(1)	X(2)	Y(1)	Y(2)
X(1)	I	0	B(1,1)	B(1,2)
X(2)	0	I	B(2,1)	B(2,2)

(a) A

	X(1)	X(2)	Y(1)	Y(2)
Y(1)	B(1,1)'			0
Y(2)	B(1,2)'			I

(b) A^*

Figure 2: Representation of M and M^*.

Consider the matrix $A^* = [B^T|I^{(n-m)\times(n-m)}]$ (Figure 2(b)). Since Z was a basis, B restricted to the X_1 rows and Y_1 columns has full rank. Thus the X_1 columns in A^* also have full rank, and so $Z^* = X_1 \cup Y_2$ is an independent set of vectors. By a similar argument, it is a maximal independent set and so is a basis. As $Z^* = S \backslash Z$, Z^* is a basis of M^*. Since this is true for every basis Z of M, A^* is a representation of the dual matroid M^* of M.

□

In 1971, after characterizations of $GF(2)$- and $GF(3)$-representable matroids, Gian-Carlo Rota conjectured that the matroids representable over any finite field can be characterized by a finite list of excluded minors (just as, for example, planar graphs can be characterized as those graphs excluding $K_{3,3}$ and K_5 as minors). A minor of a matroid M is a matroid which can be obtained from M by contractions (defined last time) and deletions of elements of the ground set.

Binary matroids, or matroids representable over $GF(2)$, were characterized by their excluded minors by Tutte in 1958 [4]. They are precisely the matroids which exclude U_4^2 as a minor. (Observe that the list of excluded minors should be closed under taking the dual, and indeed U_4^2's dual is U_4^2 itself.)

Theorem 5 *A matroid is binary if and only if it excludes U_4^2 as a minor.*

Tutte further characterized regular matroids, or matroids representable over *any* field.

Definition 1 *The Fano matroid is the matroid with ground set $S = \{A, B, C, D, E, F, G\}$ whose bases are all subsets of S of size 3 except $\{A, D, B\}$, $\{B, E, C\}$, $\{A, F, C\}$, $\{A, G, E\}$, $\{D, G, C\}$, $\{B, G, F\}$, and $\{D, E, F\}$.*

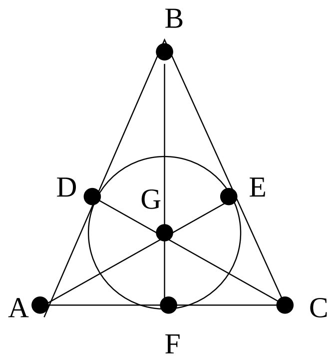


Figure 3: Fano matroids

Note Fano matroids (and hence their duals, see Theorem 4), are representable over $GF(2)$ by, for example, $A = (0, 1, 0)^T$, $B = (1, 0, 0)^T$, $C = (0, 0, 1)^T$, $D = (1, 1, 0)^T$, $E = (1, 0, 1)^T$, $F = (0, 1, 1)^T$, and $G = (1, 1, 1)^T$. In fact, these two matroids are the minimal binary non-regular matroids.

Theorem 6 *A binary matroid is regular if and only if it excludes the Fano matroid F_7 and its dual F_7^* as minors.*

The ternary matroids, or matroids representable over $GF(3)$ were characterized in the early 1970s in an unpublished work of Reid, later published by Bixby [1] and Seymour [3].

Theorem 7 *The ternary matroids are the matroids which exclude U_5^2, $U_5^{2^*} = U_5^3$, F_7, and F_7^* as minors.*

In 2000, Geelen, Gerards and Kapoor characterized matroids representable over $GF(4)$ [2] by specifying seven excluded minors, a work for which they won the 2003 Fulkerson Prize.

The current state-of-the-art is represented in Figure 4.

Remark: Linear matroids are matroids that are representable over *some* field. Not all linear matroids are representable over the rationals. The Fano matroid is an example of a binary matroid that is not representable over the rationals:

If F_7 is representable over the rationals, then it is representable over the reals. Since the basis has cardinality 3, it is representable over $\Re^3$. Assume such a representation. Since D, E, and F are dependent, they must define a plane that passes through the origin, say the xy-plane. Consider, say, D and E. Each of A, B, C, and G together with D and E form an independent set. Therefore, A, B, C, and G do not lie on the xy-plane. Thus we can project them onto the $z = 1$ plane (i.e.

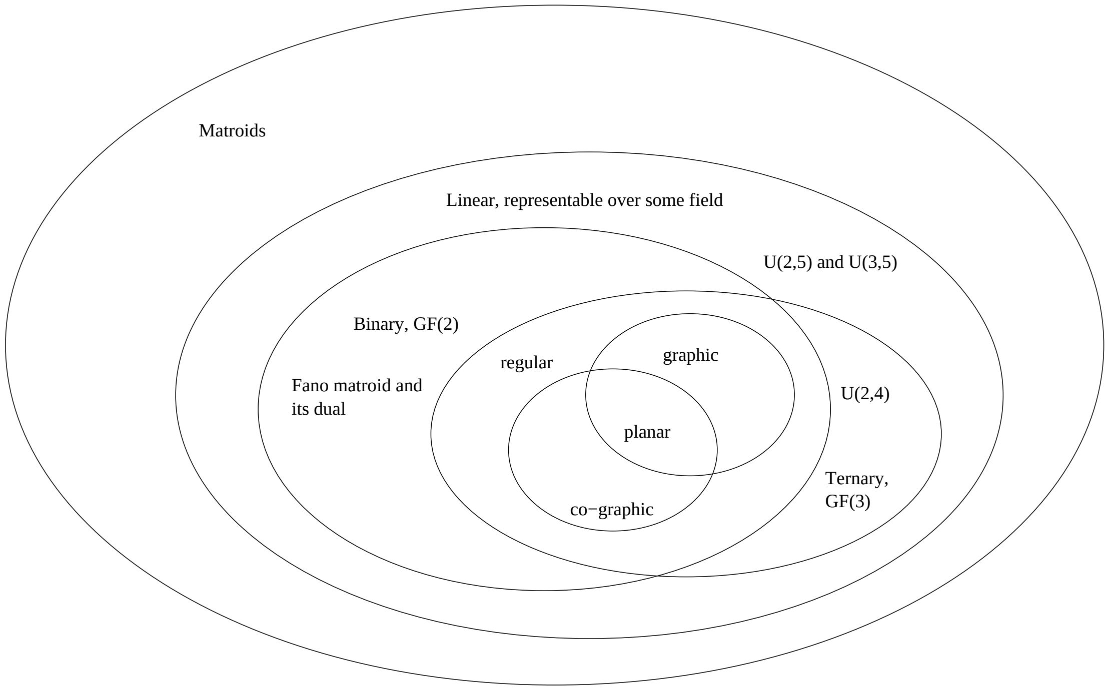


Figure 4: Classes of matroids.

scale them so that they lie in the $z = 1$ plane). This new representation, say A', B', C', and G', preserves the independence relations and thus is also a representation of F_7. Now notice that $\mathrm{span}(A, G) \cap \mathrm{span}(C, B) = \mathrm{span}(E)$. As E lies in the xy-plane, $\mathrm{span}(A, G) \cap (z = 1) = \overline{A'G'}$ and $\mathrm{span}(C, B) \cap (z = 1) = \overline{B'C'}$ form two parallel lines. Similarly, $\mathrm{span}(A, B) \cap (z = 1) = \overline{A'B'}$ and $\mathrm{span}(C, G) \cap (z = 1) = \overline{C'G'}$ form two parallel lines. Thus $A'B'C'G'$ is a parallelogram, and so its diagonals, $\overline{B'G'}$ and $\overline{A'C'}$ must intersect. However, this contradicts the fact that $\mathrm{span}(A, C) \cap \mathrm{span}(B, G) = \mathrm{span}(F)$.

3 Matroid Optimization

To show the power of matroids and just as a sample of things to come, we begin with a definition of the union of matroids. This definition will prove useful in answering questions like *does G contain k disjoint spanning trees?*

Definition 2 *The matroid union $\vee_{i=1}^k M_i$ of matroids $M_1 = (S_1, \mathcal{I}_1), \ldots, M_k = (S_k, \mathcal{I}_k)$ is the matroid $M = (\cup_{i=1}^k S_i, \mathcal{I})$ where $\mathcal{I} = \{\cup_{i=1}^k I_i : I_i \in \mathcal{I}_i\}$.*

We will show that M is a matroid; this is not completely obvious. Furthermore, one can characterize the size of a maximal independent subset in the union of matroids as follows.

Lemma 8 *Let $M = \vee_{i=1}^k M_i$ for matroids $M_i = (S_i, \mathcal{I}_i)$. Then for any $U \subseteq S$, $r_M(U) = \min_{T \subseteq U}(|U - T| + \sum_{i=1}^k r_{M_i}(T \cap S_i)$ where $r_M(U)$ is the rank of set U in matroid M.*

We will see applications of this next time. Today we will discuss a simpler optimization problem, that of finding a maximum weight independent set in a matroid. Specifically, let $M = (S, \mathcal{I})$ be a matroid with an integral weight function $w(s)$ for each $s \in S$. We would like to find an $I \in \mathcal{I}$ of maximum weight.

Consider the greedy algorithm. First order the elements of S so that $w(s_i) \geq w(s_{i+1})$. Initialize I to the empty set. At step i, if $\{s_i\} \cup I \in \mathcal{I}$, set $I \leftarrow \{s_i\} \cup I$. We will prove that this algorithm is optimal with the aid of the following polytope due to Edmonds:

Matroid Polytope

$$\begin{aligned} x_s &\geq 0 && \forall s \in S \\ x(U) &\leq r(U) && \forall U \subseteq S \end{aligned}$$

Note that the second inequality implies $x_s \leq 1$ as the rank of a single vertex is at most one. We will show that this polytope is integral and that the vertices are the indicator vectors of independent sets of the matroid. Certainly all independent sets of the matroid satisfy that $x(U) \leq r(U)$.

Theorem 9 *The greedy algorithm finds a maximum weight independent set.*

Theorem 10 *The matroid polytope of Edmonds is integral.*

Proof: (of Theorems 9 and 10) Consider the linear program

$$\begin{aligned} \max \sum_{s \in S} w(s) x_s & \\ \text{s.t.} \sum_{s \in U} x_s \leq r(U) & \qquad \forall U \subseteq S \\ x_s \geq 0 & \qquad \forall s \in S \end{aligned}$$

and its dual

$$\begin{aligned} \min \sum_{U \subseteq S} r(U) y_U & \\ \text{s.t.} \sum_{U \ni s} y_U \geq w(s) & \qquad \forall s \in S \\ y_U \geq 0 & \qquad \forall U \subseteq S. \end{aligned}$$

Consider adding the constraint that the y_U are integral to the dual. Let the optimal value of this extended dual be O'_D, the optimal value of the primal be O_P, the optimal value of the dual be O_D, and the weight of the maximum independent set be W_I. We will construct feasible y_U for the extended dual such that the value of the program O'_D will equal the weight $w(I)$ of the set I returned by the greedy algorithm. This will prove several facts. As $w(I) \leq W_I \leq O_P = O_D \leq O'_D$, this shows that the greedy algorithm is optimal, thus proving Theorem 9. Furthermore, this shows that the dual is integral for an arbitrary integral weight function, and thus the system is TDI. Together with the fact that the rank function is integral, this proves that the matroid polytope is integral, thus proving Theorem 10.

Let's prove that $O'_D = w(I)$. Label the elements of S in order of decreasing weight as the greedy algorithm does. Let $U_i = \{s_1, \ldots, s_i\}$ and set $y_{U_n} = w(s_n)$, $y_{U_i} = w(s_i) - w(s_{i+1})$ for $1 \leq i \leq n$. For all other sets U, set $y_U = 0$. Note $y_U \geq 0$ and y_U are integral by construction. The first constraint of the dual is also satisfied as, for all i, $\sum_{U \ni s_i} y_U = \sum_{j \geq i} y_{U_j} = w(s_n) + \sum_{j=i}^{n-1} w(s_j) - w(s_{j+1}) = w(s_i)$. Now consider the objective. Notice $r(U_1) = 1$ if $s_1 \in I$ and 0 otherwise. Similarly, $r(U_i) - r(U_{i-1}) = 1$ if $s_i \in I$ and 0 otherwise. Therefore,

$$O'_D = \sum_U r(U) y_U$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1} r(U_i)(w(s_i) - w(s_{i+1})) + r(U_n)w(s_n) \\
&= w(s_1)r(U_1) + \sum_{i=2}^{n} w(s_i)(r(U_i) - r(U_{i-1})) \\
&= w(I).
\end{aligned}
$$

□

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