RAA 2017
Longest Paths in Graphs: Parameterized Algorithms – Exercise I

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Remark : Several of these problems use the notion of *matroids*. Those who do not know the definition of matroids can find the definitions below. Else, assume that $\mathcal{S}$ is a family of sets over a universe U such that each set in $\mathcal{S}$ has size exactly p.

1. In the Vertex Cover problem, we are given a graph $G = (V, E)$ and a positive integer k, and the problem is to test whether there exists a vertex subset $X \subseteq V(G)$ such that $|X| \leq k$ and $G \setminus X$ is an independent set. Obtain a $2^k n^{\mathcal{O}(1)}$-time algorithm for this problem.

2. In the d-Hitting Set problem, we are given a family $\mathcal{F}$ of sets of size d over a universe U and a positive integer k, and the problem is to test whether there exists a subset $X \subseteq U$ such that $|X| \leq k$ and for every set $F \in \mathcal{F}$, $F \cap X \neq \emptyset$. Obtain a $k^{\mathcal{O}(d)}$ kernel for the problem using the method of representative sets.

3. Let $A_1, \ldots, A_m$ be p-element sets and $B_1, \ldots, B_m$ be q-element sets such that $A_i \cap B_j = \emptyset$ if and only of $i = j$.

   (a) Show that $m \leq 2^{p+q}$. (Hint: Think uniform random partition of $U = \cup_{i=1}^{m}(A_i \cup B_i)$.)

   (b) Show that $m \leq \binom{p+q}{p}$. (Hint: Think of permutations of U.)

   (c) Show that the bound of $\binom{p+q}{p}$ on m is tight.

   (d) Let $\mathcal{S} = \{S_1, \ldots, S_t\}$ be a family of p element sets. Using the above exercises show that the size of q-representative family is upper bounded by $\binom{p+q}{p}$.

4. Let $M = (U, \mathcal{I})$ be a matroid and let $\mathcal{S}$ be a p-uniform family of subsets of E. Show that if $\mathcal{S}' \subseteq_{rep}^{q} \mathcal{S}$ and $\widehat{\mathcal{S}} \subseteq_{rep}^{q} \mathcal{S}'$, then $\widehat{\mathcal{S}} \subseteq_{rep}^{q} \mathcal{S}$. (If $\widehat{\mathcal{S}} \subseteq \mathcal{S}$ is q-representative for $\mathcal{S}$ we write $\widehat{\mathcal{S}} \subseteq_{rep}^{q} \mathcal{S}$.)

5. Let $M = (U, \mathcal{I})$ be a matroid and let $\mathcal{S}$ be a p-uniform family of subsets of E. Show that if $\mathcal{S} = \mathcal{S}_1 \cup \cdots \cup \mathcal{S}_\ell$ and $\widehat{\mathcal{S}}_i \subseteq_{rep}^{q} \mathcal{S}_i$, then $\cup_{i=1}^{\ell} \widehat{\mathcal{S}}_i \subseteq_{rep}^{q} \mathcal{S}$.

6. Let G be a connected graph on $2n$ vertices and $\mathcal{L}$ be a family of forests of G of size n (that is, the number of edges is n). Let $\widehat{\mathcal{L}} \subseteq \mathcal{L}$ be a family of forests such that for any forest F of size $n-1$, if there exists a forest $X \in \mathcal{L}$ such that $F \cup X$ is a spanning tree of G, then there exists a forest $\widehat{X} \in \widehat{\mathcal{L}}$ such that $F \cup \widehat{X}$ is a spanning tree of G. Could you give a non-trivial upper bound on the size of $|\widehat{\mathcal{L}}|$ (like some c^n)?

Matroid Basics

Now we give definitions related to matroids.

Definition 1 *A pair $M = (U, \mathcal{I})$, where E is a ground set and $\mathcal{I}$ is a family of subsets (called independent sets) of E, is a* matroid *if it satisfies the following conditions:*

(I1) $\phi \in \mathcal{I}$.

(I2) *If* $A' \subseteq A$ *and* $A \in \mathcal{I}$ *then* $A' \in \mathcal{I}$.

(I3) *If* $A, B \in \mathcal{I}$ *and* $|A| < |B|$*, then there is* $e \in (B \setminus A)$ *such that* $A \cup \{e\} \in \mathcal{I}$.

The axiom (I2) is also called the hereditary property and a pair $(E, \mathcal{I})$ satisfying only (I2) is called hereditary family. An inclusion wise maximal set of $\mathcal{I}$ is called a *basis* of the matroid. Using axiom (I3) it is easy to show that all the bases of a matroid have the same size. This size is called the *rank* of the matroid M, and is denoted by $\mathsf{rank}(M)$.

Linear Matroids and Representable Matroids

Let A be a matrix over an arbitrary field $\mathbb{F}$ and let E be the set of columns of A. For A, we define matroid $M = (U, \mathcal{I})$ as follows. A set $X \subseteq E$ is independent (that is $X \in \mathcal{I}$) if the corresponding columns are linearly independent over $\mathbb{F}$. The matroids that can be defined by such a construction are called *linear matroids*, and if a matroid can be defined by a matrix A over a field $\mathbb{F}$, then we say that the matroid is representable over $\mathbb{F}$. That is, a matroid $M = (U, \mathcal{I})$ of rank d is representable over a field $\mathbb{F}$ if there exist vectors in $\mathbb{F}^d$ corresponding to the elements such that linearly independent sets of vectors correspond to independent sets of the matroid. A matroid $M = (U, \mathcal{I})$ is called *representable* or *linear* if it is representable over some field $\mathbb{F}$.

1. Show that the following families form matroid.
   (a) Let $G = (V, E)$ be a graph. Let $M = (U, \mathcal{I})$ be a matroid defined on G, where $U = E$ and $\mathcal{I}$ contains all *forests* of G. (**Graphic Matroid**)
   (b) Let $G = (V, E)$ be a connected graph. Let $M = (U, \mathcal{I})$ be a matroid defined on G, where $U = E$ and $\mathcal{I}$ contains all $E' \subseteq E$ such that $G' = (V, E \setminus E')$ is connected. (**Co-Graphic Matroid**)

2. Obtain a representation matrix for the following matroid.
   (a) Graphic Matroid.
   (b) Uniform Matroids – $M = (U, \mathcal{I})$ where $\mathcal{I}$ contains all subsets of U of size at most k for some fixed constant k.
   (c) Partition Matroids – It is defined by a ground set U being partitioned into (disjoint) sets $U_1, \ldots, U_\ell$ and by ℓ non-negative integers $k_1, \ldots, k_\ell$. A set $X \subseteq U$ is independent if and only if $|X \cap U_i| \leq k_i$ for all $i \in \{1, \ldots, \ell\}$. That is,

$$\mathcal{I} = \Big\{X \subseteq U \mid |X \cap U_i| \leq k_i,\ i \in \{1, \ldots, \ell\}\Big\}.$$

(d) Direct Sum of Matroids – Let $M_1 = (U_1, \mathcal{I}_1)$, $M_2 = (U_2, \mathcal{I}_2)$, $\cdots$, $M_t = (U_t, \mathcal{I}_t)$ be t matroids with $U_i \cap U_j = \emptyset$ for all $1 \leq i \neq j \leq t$. The direct sum $M_1 \oplus \cdots \oplus M_t$ is a matroid $M = (U, \mathcal{I})$ with $U := \bigcup_{i=1}^{t} U_i$ and $X \subseteq U$ is independent if and only if for all $i \leq t$, $X \cap U_i \in \mathcal{I}_i$.

3. Let $M_1 = (U_1, \mathcal{I}_1)$ and $M_2 = (U_2, \mathcal{I}_2)$ be two matroids such that $U = U_1 = U_2$. Define $M_1 \cap M_2$ as $M = (U, \mathcal{I})$ such that $X \in \mathcal{I}$ if and only if $X \in \mathcal{I}_1$ and $X \in \mathcal{I}_2$. Is M always a matroid? (**Matroid Intersection**)

4. Express the following as intersection of matroids (possibly more than two).

   (a) Finding a maximum matching in a bipartite graph $G = (A \cup B, E)$.

   (b) Testing whether a graph $G = (V, E)$ contains two edge disjoint spanning trees.

   (c) Finding a hamiltonian path in a directed graph $D = (V, A)$ between a pair of vertices s and t of D.