ALGEBRAIC ALGORITHMS

Goal: (A)

To find a polynomial P over field $\mathbb{F}$ such that $P \not\equiv 0$ if and only if (G,k) is a Yes instance.

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To find a polynomial P over field $\mathbb{F}$ such that $P \not\equiv 0$ if and only if (G,k) is a Yes instance.

$\boxed{\text{Of course we should be able to evaluate this polynomial efficiently.}}$

We need the notion of potential solution.

$$\updownarrow$$

These potential solution will constitute a monomial in the polynomial.

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A path on k-vertices.

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A path on k-vertices.

(unfortunately such a polynomial is hard to compute.)

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These potential solution will constitute a monomial in the polynomial.

$\updownarrow$?

Walk on k- vertices.

(Almost works but we will use "some coloring" to make computation easier as we had in "Color - Coding")

Potential Solution

Walk on k-vertices that are colored.

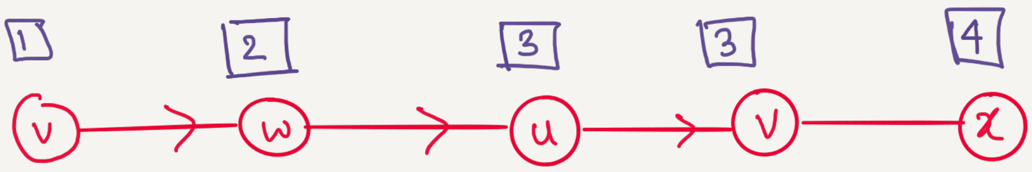
Colors that we will use will come from

$$[k] = \{1, 2, \ldots\ldots, k\}$$

Potential Solution

Walk on k-vertices that are colored.

Example:-



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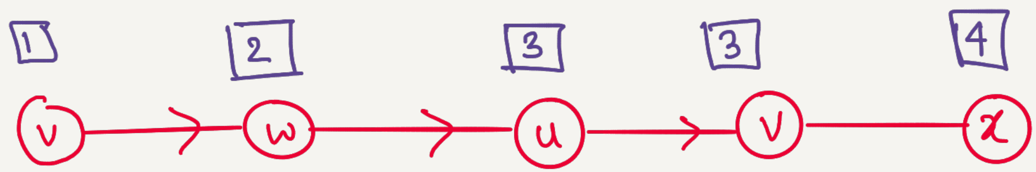
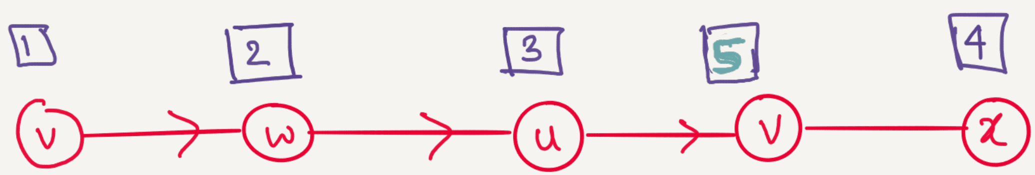


Colorful Potential Solution

Example:-



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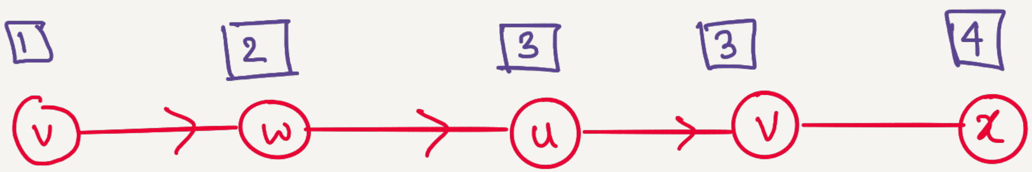

Colorful Potential Solution

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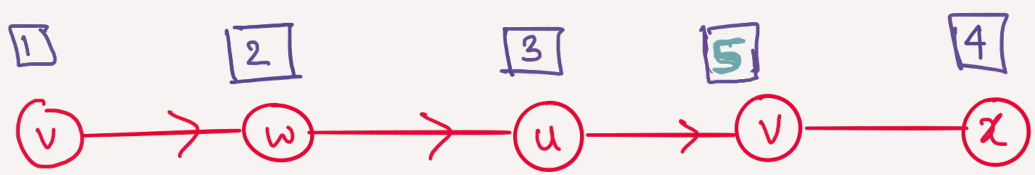


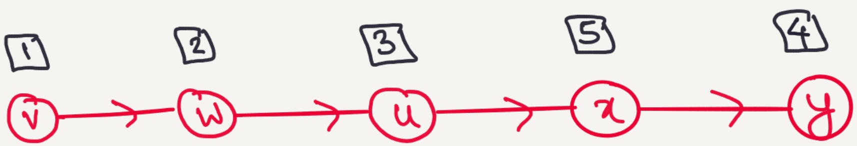
Observe that in colorful potential solution vertex can repeat.

Colorful Potential Solution (CPS)

Correct CPS

- Walk is a path

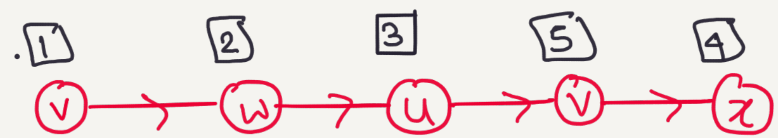
Example:-



Incorrect CPS

- Walk is not a path

Example :-



Colorful Potential Solution (CPS)

Correct CPS

— Walk is a path

Example:-

$$\begin{array}{ccccccccc} \boxed{1} & & \boxed{2} & & \boxed{3} & & \boxed{5} & & \boxed{4} \\ v & \to & w & \to & u & \to & x & \to & y \end{array}$$

all vertices are different

Incorrect CPS

— Walk is not a path

Example :-

$$\begin{array}{ccccccccc} \boxed{1} & & \boxed{2} & & \boxed{3} & & \boxed{5} & & \boxed{4} \\ v & \to & w & \to & u & \to & v & \to & x \end{array}$$

v is repeated here.

Observation

Original instance is a YES-instance
if and only if
It has a correct colorful potential solution.

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$\Rightarrow$

Observation

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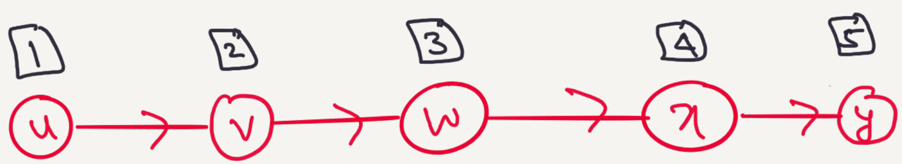
$\Leftarrow$



Let S_{col} denote the CPS.

$$P = \sum_{S \in S_{col}} \mathrm{mon}(S)$$

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What do we want :-

— Correct CPS corresponds to unique monomial in P

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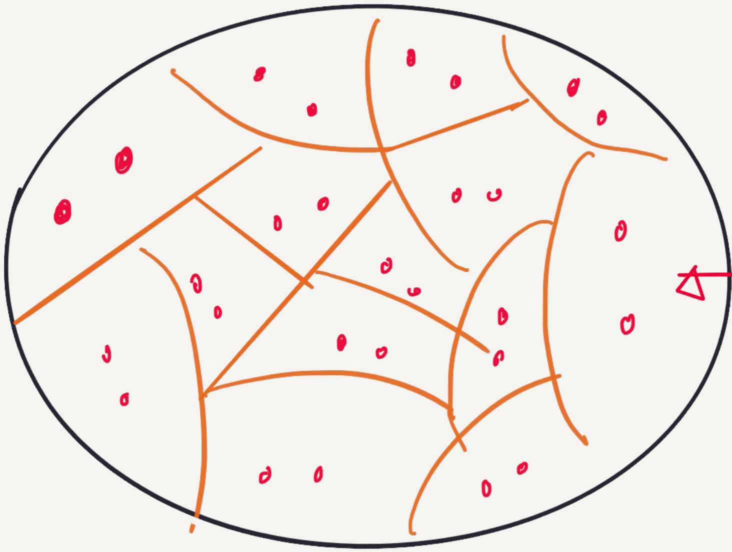
$$P = \sum_{S \in S_{col}} mon(S)$$

What do we want:-

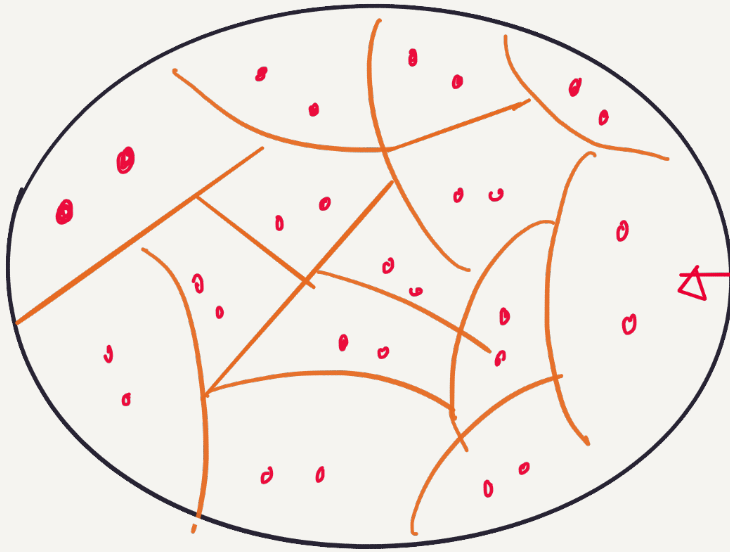
- Correct CPS corresponds to unique monomial in P
- In Correct CPS partition into pairs $\{S, T\}$ such that $mon(S) = mon(T)$

Incorrect CPS



each part has two elements and they correspond to same monomials

Incorrect CPS



each part has two elements and they correspond to same monomials

Question:- What happens if we evaluate P over a field $\mathbb{F}$ of characteristic 2.

Let S_{col} denote the CPS.

$$P = \sum_{S \in S_{col}} mon(S)$$

$$= \sum_{S \in S_{correct}} mon(S) + \sum_{S \in S_{incorrect}} mon(S)$$

Let S_{col} denote the CPS.

$$P = \sum_{S \in S_{col}} \mathrm{mon}(S)$$

$$= \sum_{S \in S_{correct}} \mathrm{mon}(S) + \cancel{\sum_{S \in S_{incorrect}} \mathrm{mon}(S)}$$

(over field of char 2)

$$= \sum_{S \in S_{correct}} \mathrm{mon}(S)$$

$$P = \sum_{S \in S_{correct}} \mathrm{mon}(S)$$

Q. 1:- How do we evaluate this?

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Did not even tell you what mon(S) is?

Monomial in Potential Solution

Vertex & Color Variables

$$X_{v,i} \qquad v \in V(G),\ i \in [k]$$

<u>Edge Variable</u>

$$Y_e \qquad e \in A(G)$$

Monomial in Potential Solution

Vertex & Color Variables

$$X_{v,i} \qquad v \in V(G),\ i \in [k]$$

Edge Variable

$$Y_e \qquad e \in A(G)$$

Given a walk W, $col : [k] \to [k]$, $f : [k] \to V(G)$

we will have a monomial.

$col(i)$ represents the color of i^{th} vertex of W | $f(i)$ represents the i^{th} vertex of W

Example

W

$$x \longrightarrow u \longrightarrow v \longrightarrow w$$

$col : [4] \to [4]$

$col(1) = 1, \quad col(2) = 3, \quad col(3) = 4, \quad col(4) = 2$

$f : [4] \to V(G)$

$f(1) = x, \quad f(2) = u, \quad f(3) = v, \quad f(4) = w$

Monomial corresponding to W, col, f will be

$$\left(\prod_{i=1}^{k} X_{f(i), col(i)} \right) \cdot \left(\prod_{i=1}^{k-1} y_{f(i)\, f(i+1)} \right)$$

ex:- $$\left(X_{x,1} \cdot X_{u,3} \cdot X_{v,4} \cdot X_{w,2} \right) \cdot \left(Y_{xu} \cdot Y_{uv} \cdot Y_{vw} \right)$$

Let S_{col} denote the CPS.

$$P = \sum_{S \in S_{col}} \text{mon}(S)$$

What do we want :-

— Correct CPS corresponds to unique monomial in P

(B) In Correct CPS partition into pairs $\{S, T\}$ such that $\text{mon}(S) = \text{mon}(T)$

Let us show (B) first (Monomials corresponding to Incorrect CPS cancel out)

Let $W = v_1, \ldots, v_k$ be a walk, col, f

$$\mathrm{col} : [r] \longrightarrow [k], \quad f : [k] \longrightarrow V(G)$$

$$f(i) = v_i$$

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Question:-

What do we know about W?

W is a walk and hence there exist indexes i & j such that

$$v_i = f(i) = f(j) = v_j \quad \& \quad i < j$$

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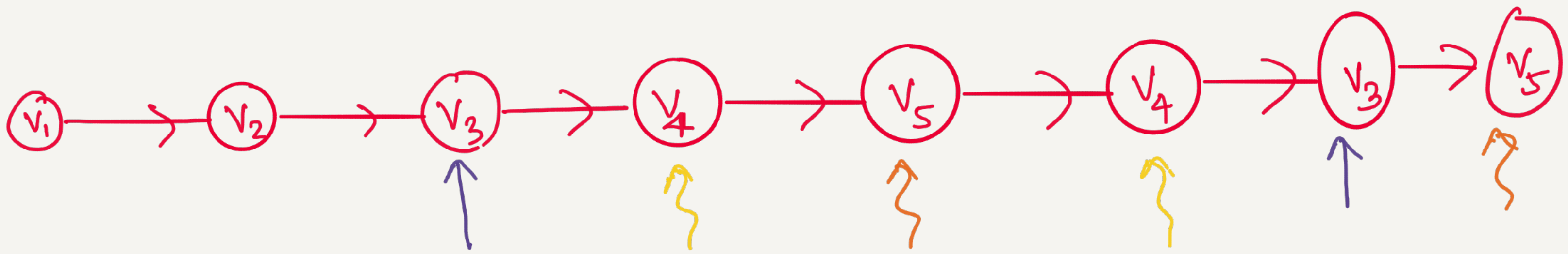
Among such pairs choose the

$\boxed{\text{lexicogoaphically first pair } (i,j).}$

Example

$$v_1 \to v_2 \to v_3 \to v_4 \to v_5 \to v_4 \to v_3 \to v_5$$

Example



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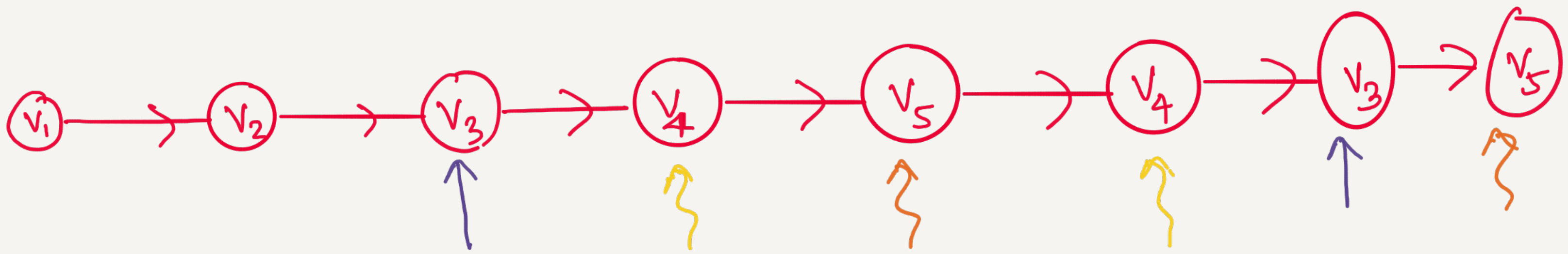

So the indexes that repeat are:-

$$(3, 7), (4, 6), (5, 8)$$

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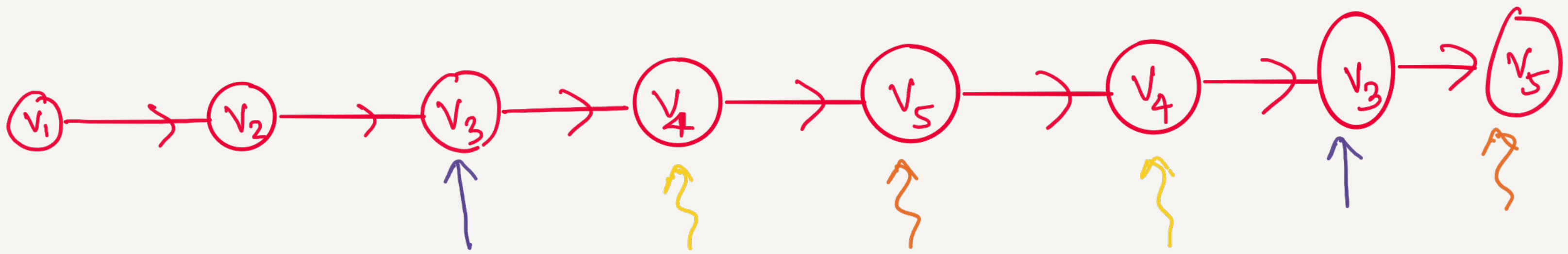

So the indexes that repeat are:-

$$(3, 7), \ (4, 6), \ (5, 8)$$

Answer is:- $(3,7)$

→ (essentially — first index whose vertex repeats, the index on which this vertex occurs second time)

$\phi : S_{incorrect} \longrightarrow S_{incorrect}$

$(W, col, f) \longrightarrow (W, \widetilde{col}, f)$

Let (i, j) be the lexicographically first pair of W that repeats.

$$\widetilde{col}(x) = \begin{cases} col(j) & \text{if } x = i \\ col(i) & \text{if } x = j \\ col(x) & \text{otherwise} \end{cases}$$

"swap the color of vertices on index i & j"

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$\boxed{\cdot}$ $(W, col, f) \neq (W, \widetilde{col}, f)$ $\because$ col is injective/colorful.

$\boxed{\cdot}$ $mon(W, col, f) = \left(\prod_{t=1}^{k} X_{f(t), col(t)} \right) \cdot \underbrace{\left(\prod_{t=1}^{k-1} Y_{f(t) f(t+1)} \right)}_{Z}$

$= \left(\prod_{[k] \setminus \{i,j\}} X_{f(t), col(t)} \right) \underbrace{X_{f(i), col(i)}}_{X_{f(i), \widetilde{col}(i)}} \cdot \underbrace{X_{f(j), col(j)}}_{X_{f(j), \widetilde{col}(j)}} \cdot Z = mon(W, \widetilde{col}, f)$

- Observe that

$$(W, \omega, f) \longrightarrow (W, \widetilde{\omega}, f)$$

What is: $\phi((W, \widetilde{\omega}, f)) = ?$

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Look at the $\overrightarrow{\text{assignment}}$ process

W remains the same index (i,j) remains the same.

$\widetilde{col}$ differs from col only at i & j & that also flips the color of i & j.

- Observe that

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Look at the $\overrightarrow{\text{assignment}}$ process | W remains the same index (i, j) remains the same.

| $\widetilde{col}$ differs from col only at i & j & that also flips the color of i & j.

$$\Rightarrow \phi((W, \widetilde{col}, f)) = (W, col, f)$$

$$\phi : S_{incorrect} \longrightarrow S_{incorrect}$$

$$(W, \omega l, f) \longrightarrow (W, \widetilde{\omega l}, f)$$

- $(W, \omega l, f) \neq (W, \widetilde{\omega l}, f)$
- $\phi(\phi((W, \omega l, f))) = \phi((W, \widetilde{\omega l}, f))$

$$= (W, \omega l, f)$$

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$$\boxed{\text{Such } \phi \text{ is called fixed-point free involution.}}$$

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$$(W, \omega l, f) \longrightarrow (W, \widetilde{\omega l}, f)$$

- $(W, \omega l, f) \neq (W, \widetilde{\omega l}, f)$
- $\phi(\phi((W, \omega l, f))) = \phi((W, \widetilde{\omega l}, f))$
$$= (W, \omega l, f)$$

- Since the field $\mathbb{F}$ is of characteristic 2,
  $mon(W, \omega l, f)$ and $mon(W, \widetilde{\omega l}, f)$
  cancels out.

If $\mathbf{P} \not\equiv 0$ then there is a k-PATH

Let us look at P

$$P = \sum_{S \in S_{\omega 1}} \mathrm{mon}(S) = \sum_{(w, \omega 1, f) \in S_{\omega 1}} \mathrm{mon}(w, \omega 1, f)$$

$$= \sum_{(w, \omega 1, f) \in S_{\omega 1}} \left(\prod_{t=1}^{k} X_{f(t), \omega 1(t)} \right) \cdot \left(\prod_{t=1}^{k-1} Y_{f(t) f(t+1)} \right)$$

Let us look at P

$$P = \sum_{S \in S_{\omega l}} \mathrm{mon}(S) = \sum_{(w, \omega l, f) \in S_{\omega l}} \mathrm{mon}(w, \omega l, f)$$

$$= \sum_{(w, \omega l, f) \in S_{\omega l}} \underbrace{\left(\prod_{t=1}^{k} X_{f(t), \omega l(t)}\right)}_{A} \cdot \underbrace{\left(\prod_{t=1}^{k-1} Y_{f(t) f(t+1)}\right)}_{B}$$

Question:- Why do we exactly need A·B?
Why not only A?

Let us look at P

$$P = \sum_{S \in S_{col}} \text{mon}(S) = \sum_{(w,col,f) \in S_{col}} \text{mon}(w, col, f)$$

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<u>Question</u>:- Why do we exactly need A·B?

Why not only A?

$$\begin{array}{ccccccccc} \boxed{1} & & \boxed{2} & & \boxed{3} & & \boxed{5} & & \boxed{4} \\ a & \to & b & \to & c & \to & d & \to & e \end{array}$$

$$X_{a,1} \cdot X_{b,2} \cdot X_{c,3} \cdot X_{d,5} \cdot X_{e,4}$$

Let us look at P

$$P = \sum_{S \in S_{col}} mon(S) = \sum_{(w, col, f) \in S_{col}} mon(w, col, f)$$

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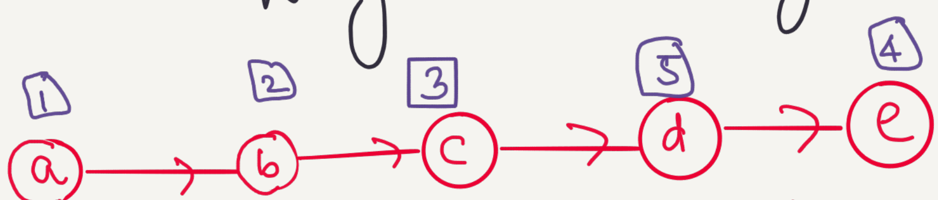
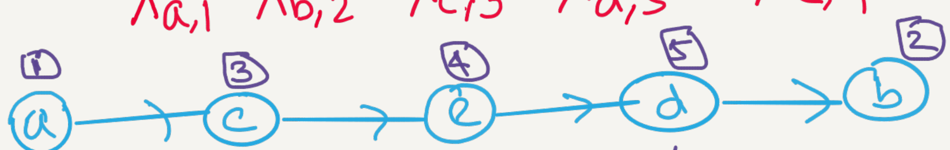
<u>Question</u>:- Why do we exactly need $A \cdot B$?
Why not only A?



$X_{a,1} \cdot X_{b,2} \cdot X_{c,3} \cdot X_{d,5} \cdot X_{e,4}$



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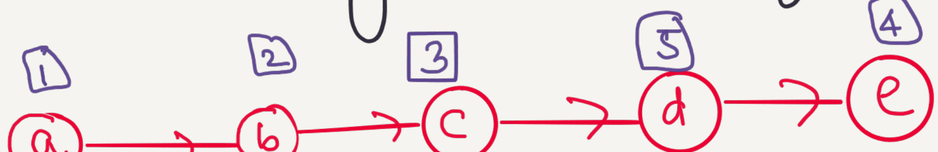

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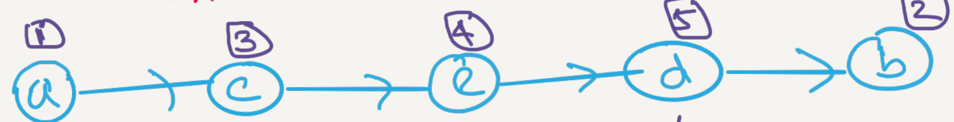

$$X_{a,1} \cdot X_{c,3} \cdot X_{e,4} \cdot X_{d,5} \cdot X_{b,2}$$

Two distinct $\boxed{\text{paths}}$ get same monomials.

Let us look at P

$$P = \sum_{S \in S_{col}} \mathrm{mon}(S) = \sum_{(W, col, f) \in S_{col}} \mathrm{mon}(W, col, f)$$

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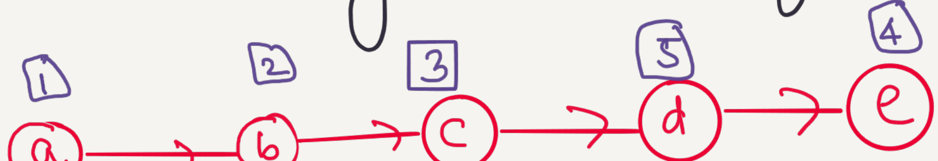
Question:- Why do we exactly need A·B? Why not only A?



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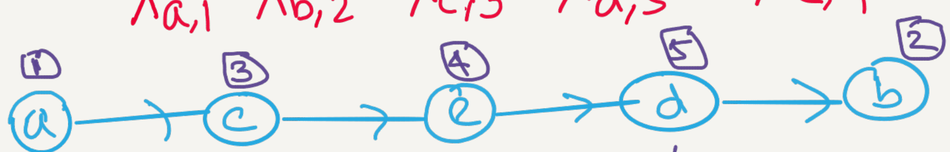

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Two distinct $\boxed{\text{paths}}$ get same monomials.

Over $\mathbb{F}$ of char 2 they can cancel out.

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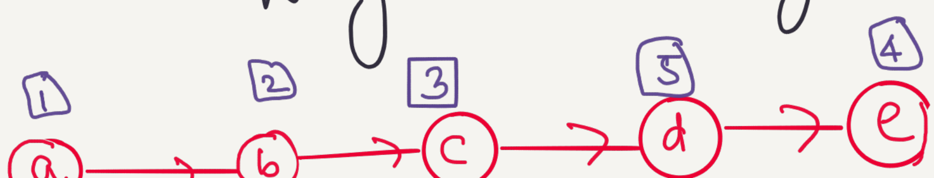
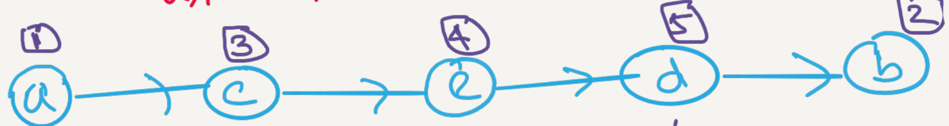


$$X_{a,1} \cdot X_{b,2} \cdot X_{c,3} \cdot X_{d,5} \cdot X_{e,4} \cdot Y_{ab} \cdot Y_{bc} \cdot Y_{cd} \cdot Y_{de}$$



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$$P = \sum_{S \in S_{col}} mon(S) = \sum_{(w, col, f) \in S_{col}} mon(w, col, f)$$

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Question:- Why do we exactly need A·B?
~~Why not only A?~~

Answer:- Every correct CPS (CPS that is a path) gets a unique monomial.

If $P \not\equiv 0$ then there is a k-path in G.

If there is a k-path in G then $P \not\equiv 0$.

$P \neq 0$ iff there is a k-path in G.

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  How do we evaluate P? (Will see later)

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However to evaluate use Schwartz-Zippel Lemma.

- $\deg(\mathbf{P}) = 2k-1 = d$
- randomly assign elements from field $\mathbb{F}$ to variables
- $\Pr(\mathbf{P}(\ ,\ ,\ ,\ ) \neq 0) \geq \left(1 - \frac{d}{|\mathbb{F}|}\right)$

given that $P \not\equiv 0$

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- $\deg(\mathbf{P}) = 2k-1 = d$
- randomly assign elements from field $\mathbb{F}$ to variables
- $\Pr(\mathbf{P}(\,,\,,\,,\,) \neq 0) \geq \left(1 - \frac{d}{|\mathbb{F}|}\right)$

given that $P \not\equiv 0$

- Choose $|\mathbb{F}| \geq 10d$, $\mathbb{F} = 2^{\lceil \log_2 10d \rceil}$

If (G, K) is a No-instance, then we get No with probability 1.

If (G, K) is a Yes-instance, then we get Yes with probability $9/10$.

Evaluation of $\mathbf{P}(\ ,\ ,\ ,\ ,\ )$.

$$P = \sum_{T \in S_{col}} mon(T)$$

S_L = potential solution that uses only colors from L

$L \subseteq [k]$

- Now potential solution may not be colorful.
- A potential solution may not use all the colors in L.

Evaluation of $\mathbf{P}(\ ,\ ,\ ,\ ,\ )$.

$$P = \sum_{T \in S_{col}} \text{mon}(T)$$

S_L = potential solution that uses only colors from L

$L \subseteq [k]$

$$S_L = \left\{ (W, col, f) \;\middle|\; \begin{array}{l} W \text{ on } k \text{ vertices } \& \\ col : [k] \longrightarrow L \end{array} \right\}.$$

$$P_L \triangleq \sum_{T \in S_L} \text{mon}(T)$$

Will show, over a field $\mathbb{F}$ of char 2

$$\mathbf{P} \stackrel{?}{=} \sum_{L \subseteq [k]} P_L = \sum_{L \subseteq [k]} \sum_{T' \in S_2} \operatorname{mon}(T')$$

$$\mathbf{P} = \sum_{T \in S_{col}} \operatorname{mon}(T)$$

Will show, over a field $\mathbb{F}$ of char 2

$$P \stackrel{?}{=} \sum_{L \subseteq [k]} P_L = \sum_{L \subseteq [k]} \sum_{T' \in S_L} \mathrm{mon}(T')$$

$$\| $$

$$\sum_{T \in S_{col}} \mathrm{mon}(T)$$

- If $T \in S_{col}$ then it is counted only once in LHS. For $L = [k]$.

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$$P = \sum_{T \in S_{col}} \operatorname{mon}(T)$$

- If $T \in S_{col}$ then it is counted only once in LHS. For $L = [k]$.
- Let S_{all} be all potential solution. & $T^* \in S_{all} \setminus S_{col}$

$T^* = (W, col, f)$ such that $col(W) \subsetneq [k]$.

Will show T^* occurs even # of times in LHS

Will show, over a field $\mathbb{F}$ of char 2

$$P \overset{?}{=} \sum_{L \subseteq [k]} P_L = \sum_{L \subseteq [k]} \sum_{T' \in S_L} \mathrm{mon}(T')$$

$$\| \quad \sum_{T \in S_{col}} \mathrm{mon}(T)$$

- If $T \in S_{col}$ then it is counted only once in LHS. For $L = [k]$.

- Let S_{all} be all potential solution.
  & $T^* \in S_{all} \setminus S_{col}$
  $T^* = (W, col, f)$ such that $\underset{Q}{\underbrace{col(W)}} \subsetneq [k]$.

for every $\tilde{Q} \supseteq Q$, T^* is in $S_{\tilde{Q}}$.

$\Rightarrow$ # of $\tilde{Q}$ is $2^{k-|Q|} = 2^a$, $a \geq 1$ & hence even.

Will show, over a field $\mathbb{F}$ of char 2

$$\mathbf{P} = \sum_{L \subseteq [k]} P_L = \sum_{L \subseteq [k]} \sum_{T' \in S_L} \text{mon}(T')$$

$$||$$

$$\sum_{T \in S_{col}} \text{mon}(T)$$ (So using inclusion-exclusion we have established this formula).

Will show how to evaluate

$P_L \leftarrow$ in polynomial time & polynomial space

thus since $\# L$ is 2^k we get $O^*(2^k)$ algorithm.

Evaluation of $\mathbf{P}(\,,\,,\,,\,)$.

$$P = \sum_{T \in S_{col}} \mathrm{mon}(T)$$

S_L = potential solution that uses only colors from L

$L \subseteq [k]$

$$S_L = \left\{ (W, col, f) \;\middle|\; \begin{array}{l} W \text{ on } k \text{ vertices } \& \\ col : [k] \longrightarrow L \end{array} \right\}.$$

reduces to evaluating

$$P_L \triangleq \sum_{T \in S_L} \mathrm{mon}(T)$$

Will do dynamic programming to compute $P_L(\,,\,,\,\cdots,\,)$

$M[v, \ell] \leftarrow$ evaluation of the polynomial

$$\sum_{S} \text{mon}(S)$$

$S = (W, \text{col}, f)$, where W is a walk on ℓ vertices that ends at vertex v and $\text{col} : [\ell] \to L$

$M[v, \ell]$ ← evaluation of the polynomial

$$\sum mon(S)$$

$S = (W, col, f)$, where W is a walk on ℓ vertices that ends at vertex v and $col : [\ell] \to L$

$$M[v, \ell] = \sum_{\substack{u \in N^-(v), \\ i \in L}} M[u, v] \cdot Y_{uv} \cdot X_{v,i}$$

in-neighbors

color assigned to the vertex v, which can be any color in L.

$$= \left(\sum_{u \in N^-(v)} M[u, v] \cdot Y_{uv} \right) \cdot \left(\sum_{i \in L} X_{v,i} \right)$$

$M[v,\ell] \leftarrow$ evaluation of the polynomial

$$\sum \text{mon}(s)$$

$s = (W, col, f)$, where W is a walk on ℓ vertices that ends at vertex v and $col: [\ell] \to L$

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in-neighbors

color assigned to the vertex v, which can be any color in L.

$$= \left(\sum_{u \in N^-(v)} M[u,v] \cdot Y_{uv} \right) \cdot \left(\sum_{i \in L} X_{v,i} \right)$$

(clearly this can be computed in polynomial time.)

ANY QUESTIONS?

Question:- Does this work for undirected graphs?

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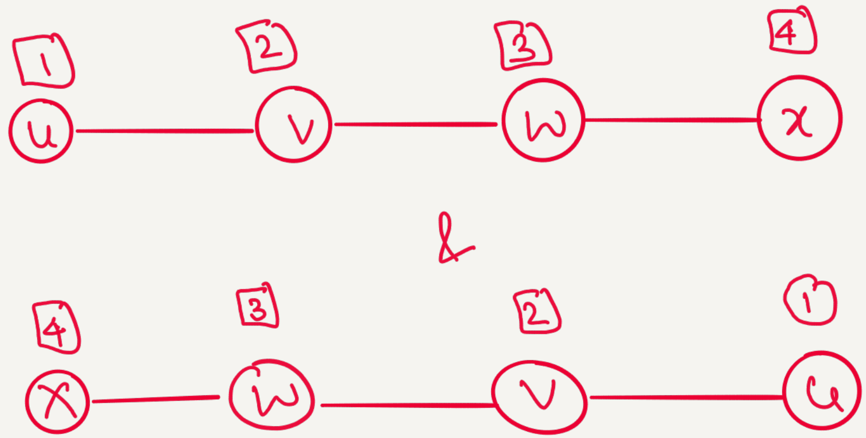
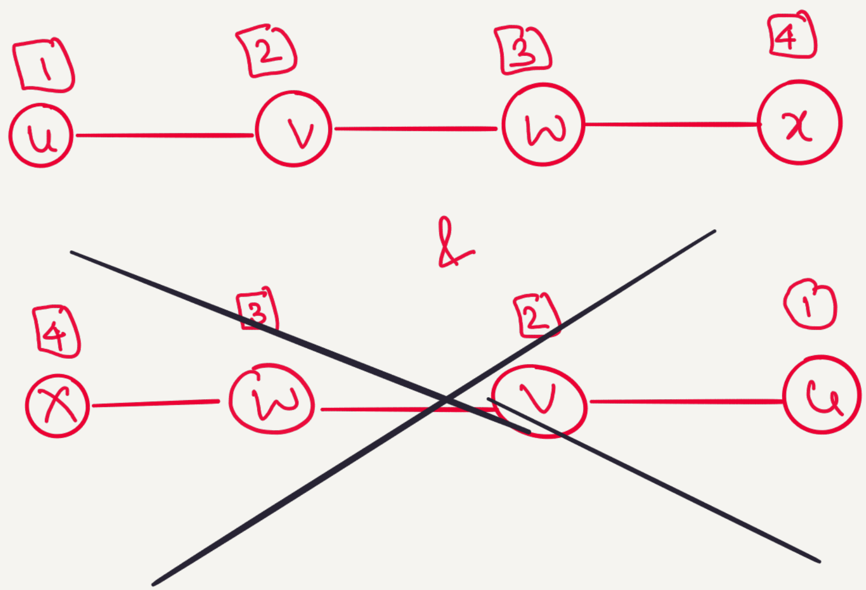
<u>Answer</u>:- No.



these path & their reverse will cancel out!

Question:- Does this work for undirected graphs?

Answer:- No.



Fix

Fix a vertex v & only look for paths starting at a vertex v! say u.

Problem 2:-

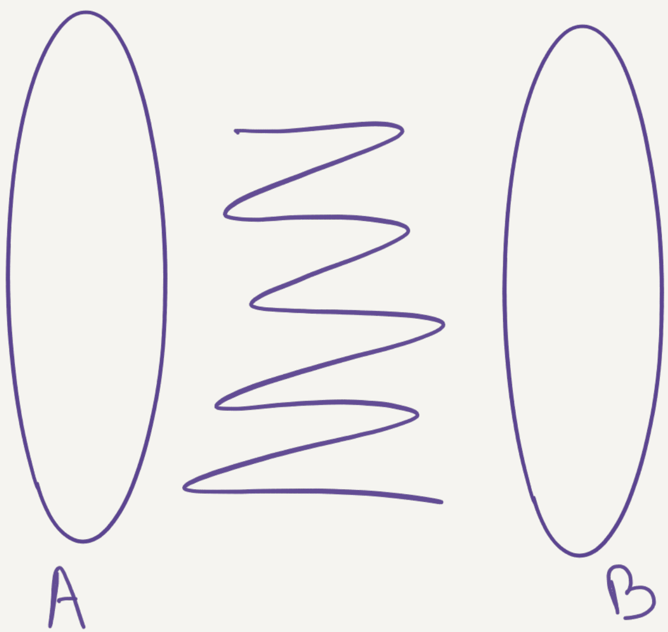
$O^*\left(2^{k/2}\right)$ algorithm for k-path in an undirected bipartite graphs.

Everything remains same but the polynomial changes.

(*) Only focus on colorful potential solution as using simple inclusion - exclusion as before we can reduce potential solution to the case of colorful potential solution case!

Everything remains same but the polynomial changes.

* Only focus on colorful potential solution



Variables

$\forall e \in E(G)$

Y_e

$\forall v \in A, \ i \in [k/2]$

$X_{v,i}$

⊛ Only focus on colorful potential solution

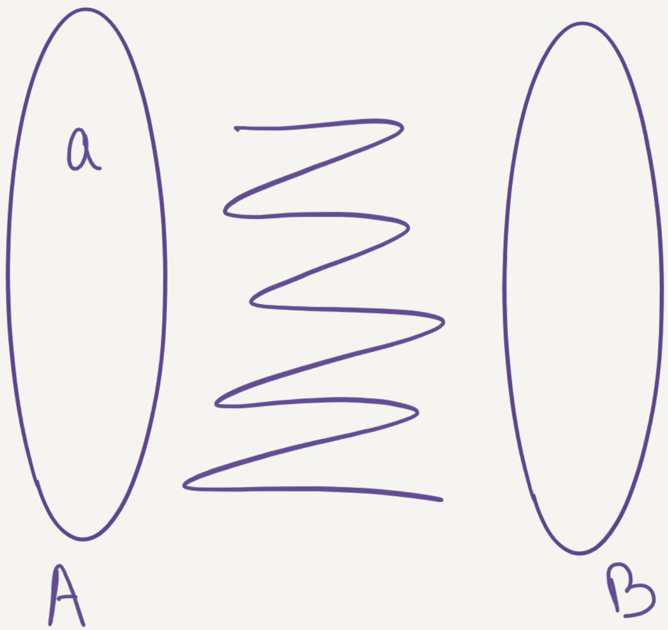
$\hat{A} = \{1, 3, 5, \dots\}$

(W, col, f)

- $col : \hat{A} \to [\frac{k}{2}]$
- $f : [k] \to V(G)$
- W is a walk on k- vertices that has no "digon"

$$\begin{array}{ccccc} V & - & U & - & V \\ \uparrow & & \uparrow & & \uparrow \\ B & & A & & B. \end{array}$$

this assumption is okay.



Variables

$\forall e \in E(G)$

Y_e

$\forall v \in A,\ i \in [k/2]$

$X_{v,i}$

Some assumption
- first vertex is $a \in A$
- k is even.

⊛ Only focus on colorful potential solution

(W, col, f)

- $col : \hat{A} \to [\frac{k}{2}]$
- $f : [k] \to V(G)$
- W is a walk on k-vertices that has no "digon"

$$\begin{array}{ccccc} x & - & y & - & x \\ \uparrow & & \uparrow & & \uparrow \\ B & & A & & B. \end{array}$$

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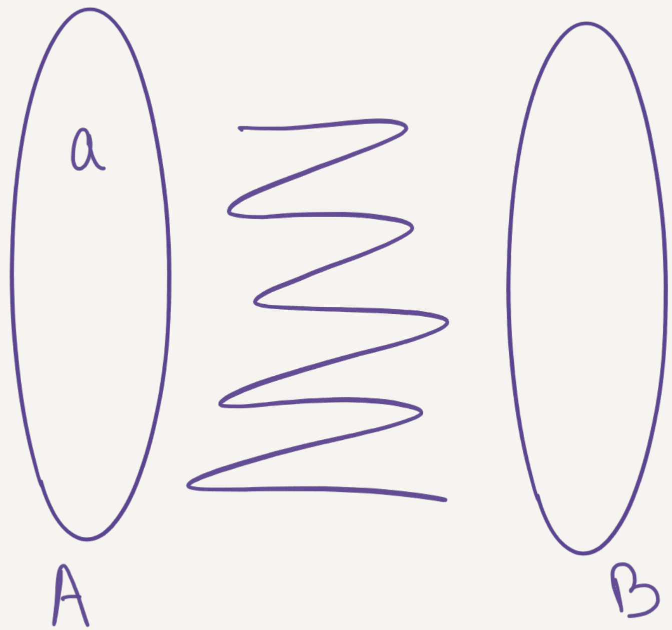


Variables

$\forall e \in E(G)$
Y_e

$\forall v \in A,\ i \in [k/2]$
$X_{v,i}$

Some assumption

- first vertex is $a \in A$
- k is even.

$M[\underline{vu}, l]$

Here the walk ends at v and the vertex before v is u.

$$M[vu, l] = \sum_{\substack{w \in N^-(u) \\ \setminus \{v\}}} M[uw, l-1] \ldots\ldots\ldots .$$

Now if $v \in B$

So this way we can assure that all the walks we consider do not have digons.

$$M[vu, l] = \sum_{w \in N^-(u) \setminus \{v\}} M[uw, l-1] \ldots\ldots$$

Now if $v \in B$

So this way we can assure that all the walks we consider do not have digons.

→ Infact we can ensure that no particular constant size pattern exists.

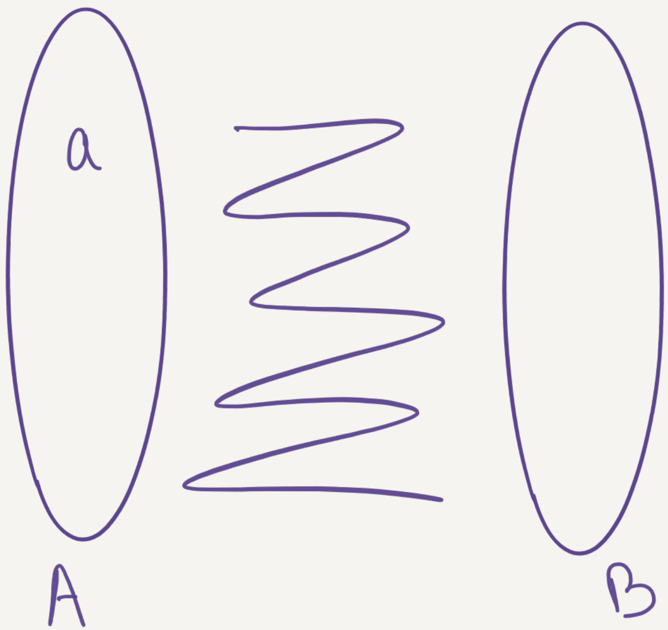
⊛ Only focus on colorful potential solution

(W, col, f)

- $col: \hat{A} \to [\frac{k}{2}]$
- $f: [k] \to V(G)$
- W is a walk on k-vertices that has no "digon"

$$\begin{matrix} V & - & U & - & V \\ \Uparrow & & \Uparrow & & \Uparrow \\ B & & A & & B. \end{matrix}$$

this assumption is okay.



Variables

$\forall e \in E(G)$

Y_e

$\forall v \in A,\ i \in [k/2]$

$X_{v,i}$

Some assumption

- first vertex is $a \in A$
- k is even.

(W, col, f)

$$mon(W, col, f) = \left(\prod_{\substack{i \in [k] \\ i \text{ odd}}} X_{f(i), col(i)} \right) \cdot \left(\prod_{i \in [k-1]} Y_{f(i) f(i+1)} \right)$$

⊛ Only focus on colorful potential solution

(W, col, f)

- $col : \hat{A} \to [\frac{k}{2}]$
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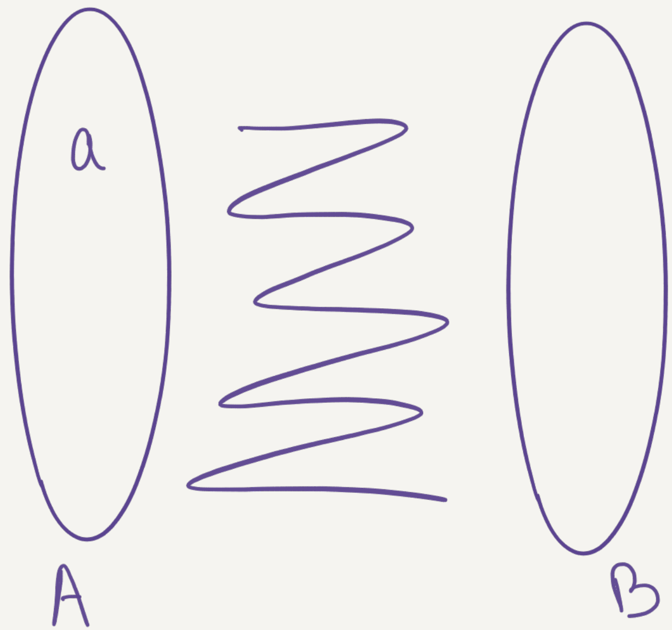

Variables

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$\forall v \in A,\ i \in [k/2]$
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Some assumption
- first vertex is $a \in A$
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(W, col, f)

$$= \left(\prod_{\substack{i \in [k] \\ i \text{ is odd}}} x_{f(i), col(i)} \right) \cdot \left(\prod_{i \in [k-1]} x_{f(i) f(i+1)} \right)$$

$\phi : S_{correct} \longrightarrow S_{correct}$

- $(W, col, f) \longrightarrow (W, \widetilde{col}, f)$

As before if there exist a vertex in A that repeats.

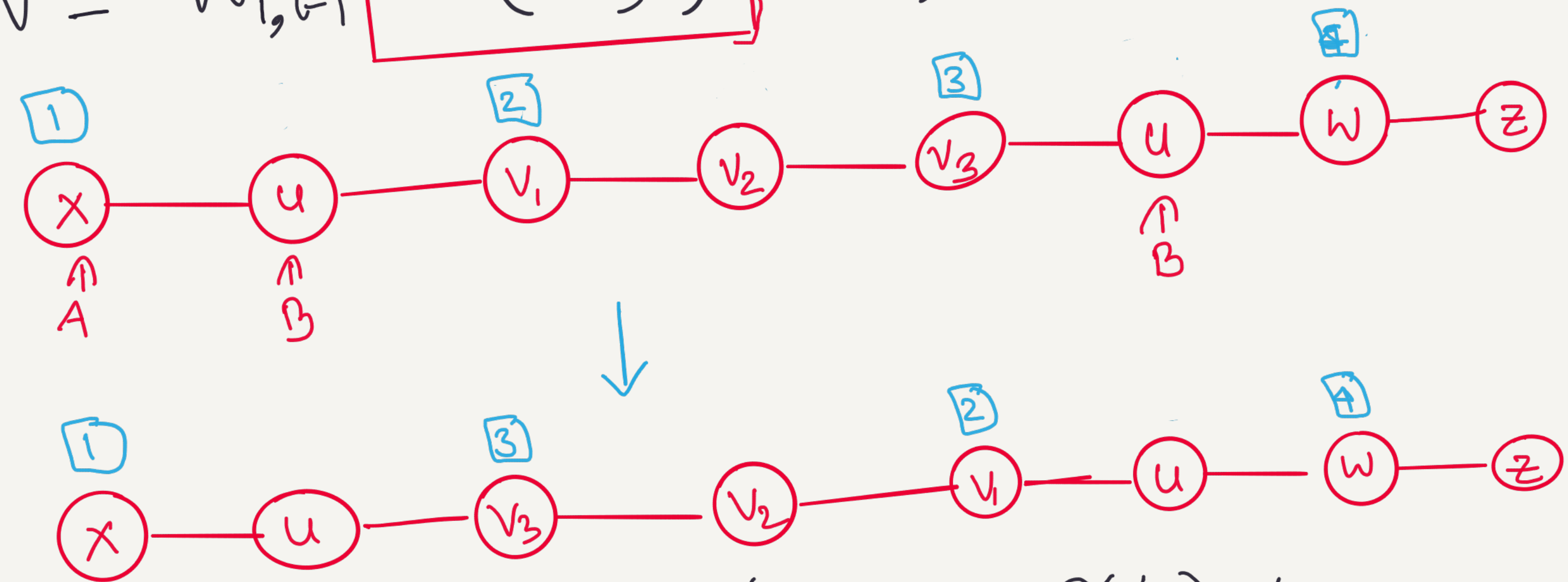
Next we handle the case when no vertices of A repeats. in W

Let (i,j) be the lexicographically first pair of indices where the vertices of B repeats.

$$W = W_{1,i-1}\, W_{i,j}\, W_{j+1,k}$$

$$\widetilde{W} = W_{1,i-1}\, \boxed{rev(W_{i,j})}\, W_{j+1,k}$$

Accordingly we have $\widetilde{col}$, $\tilde{f}$

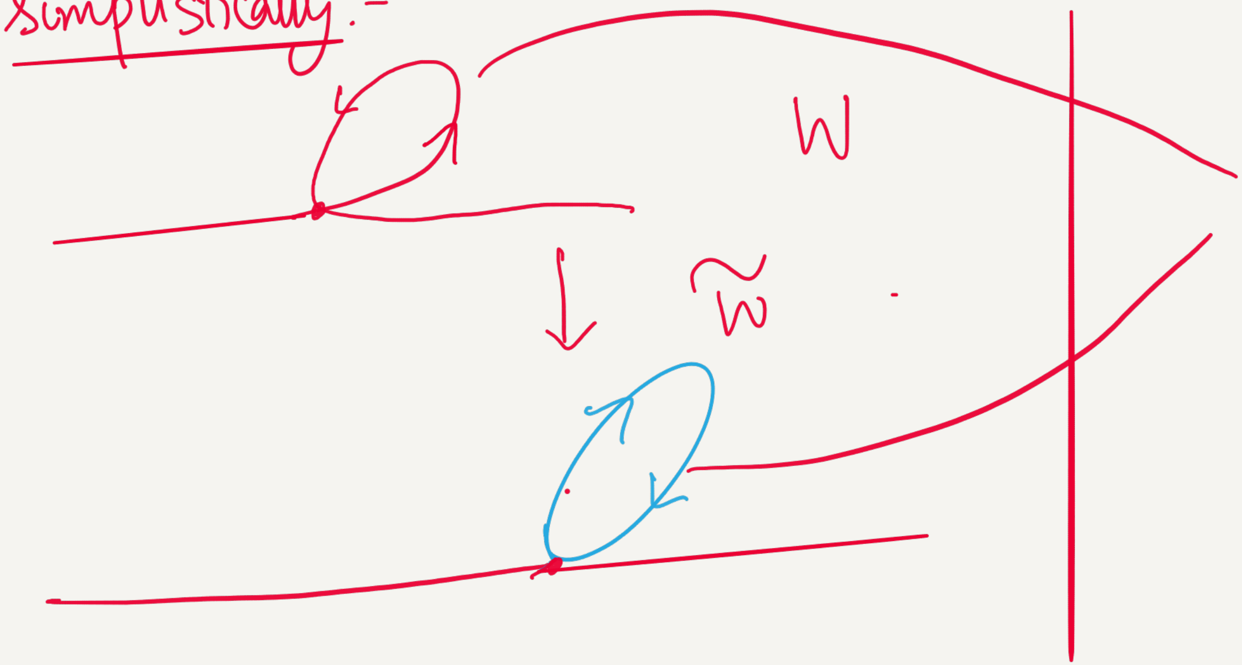


$\widetilde{col}(1)=1$, $\widetilde{col}(3)=3$, $\widetilde{col}(5)=2$, $\widetilde{col}(7)=4$

$\tilde{f}(1)=x$, $\tilde{f}(2)=u$, $\tilde{f}(3)=v_3$, $\tilde{f}(4)=v_2$, $\tilde{f}(5)=v_1$, $\tilde{f}(6)=u$, $\tilde{f}(7)=w$, $\tilde{f}(8)=z$

$$\phi((W, \omega l, f)) \text{ — } (\tilde{W}, \tilde{\omega l}, \tilde{f})$$

Simplistically:-



these may also have repeated vertices.

ϕ is a fixed point free involution.

- if vertices of A repeats clearly it is fixed point free because col is injective.
- So assume no vertex of A repeats

Assume $\phi(S) = S$

ϕ is a fixed point free involution.

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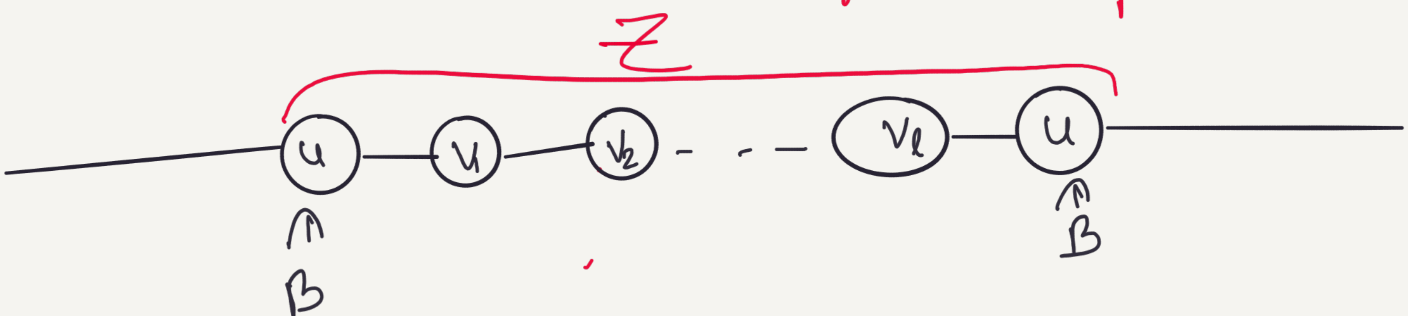


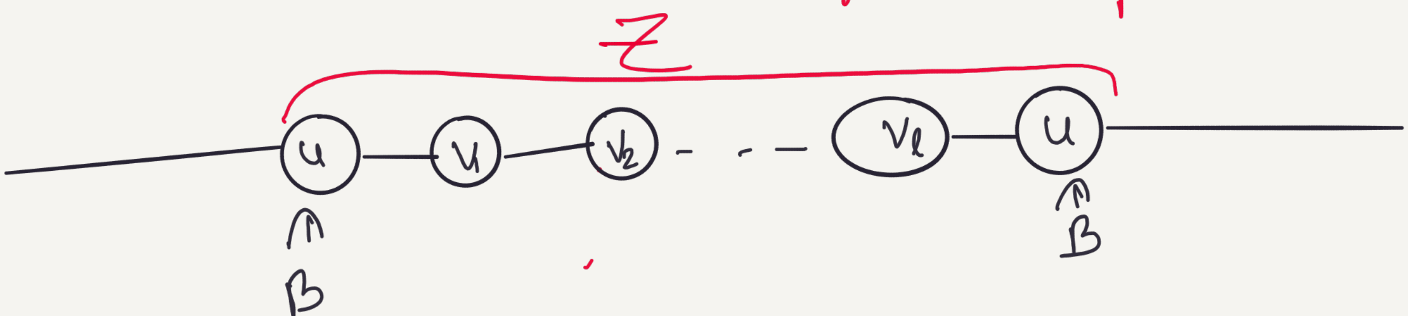
Assume $\phi(S) = S$

then Z is palindrome. $\Rightarrow$ No then the vertex of A will repeat

Can, $l \geq 3$

ϕ is a fixed point free involution. ✓

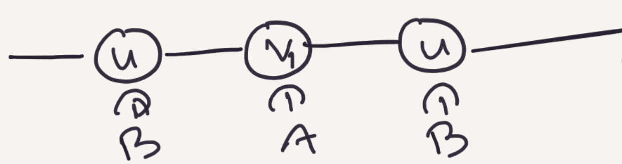
- if vertices of A repeats clearly it is fixed point free because col is injective.
- So assume no vertex of A repeats



Assume $\phi(S) = S$

then Z is palindrome. $\not\Rightarrow$ No then the vertex of A will repeat.

Can, $l \geq 3$

If $l=1$, then  is a "digon".

ϕ is a fixed point free involution.

- if vertices of A repeats clearly it is involution. (as before)
- So assume no vertex of A repeats, now lexicographically pair of indices ensure that we get involution again.

$$\phi : S_{incorrect} \longrightarrow S_{incorrect}$$
$$(w, wl, f) \longrightarrow (\tilde{w}, \widetilde{wl}, \tilde{f})$$

- $(w, wl, f) \neq (\tilde{w}, \widetilde{wl}, \tilde{f})$
- $\phi(\phi((w, wl, f))) = \phi((\tilde{w}, \widetilde{wl}, \tilde{f}))$
  $= (w, wl, f)$

- Since the field $\mathbb{F}$ is of characteristic 2, $mon(w, wl, f)$ and $mon(\tilde{w}, \widetilde{wl}, \tilde{f})$ cancels out.

If $\mathbf{P} \not\equiv 0$ then there is a k-PATH

Let us look at P

$$P = \sum_{S \in S_{col}} mon(S) = \sum_{(w, col, f) \in S_{col}} mon(w, col, f)$$

$$= \sum_{(w,col,f)\in S_{col}} \underbrace{\left(\prod_{\substack{t \in [k],\\ t \text{ is odd}}} X_{f(t), col(t)} \right)}_{Z_1} \cdot \underbrace{\left(\prod_{t=1}^{k-1} Y_{f(t) f(t+1)} \right)}_{Z_2}$$

Question:- Why do we exactly need $Z_1 \cdot Z_2$?
~~Why not only Z_1?~~

Answer:- Every correct CPS (CPS that is a path) gets a $\boxed{\text{unique monomial}}$.

If $\mathbf{P} \not\equiv 0$ then there is a k-path in G.

If there is a k-path in G then $\mathbf{P} \not\equiv 0$.

$P \neq 0$ iff there is a k-path in G.

Evaluation of $\mathbf{P}(\,,\,,\,,\,)$.

$$P = \sum_{T \in S_{col}} mon(T)$$

S_L = potential solution that uses only colors from L

$L \subseteq [k/2]$

$$S_L = \left\{ (W, col, f) \;\middle|\; \begin{array}{l} W \text{ on } k \text{ vertices } \& \\ col : [\hat{A}] \longrightarrow L \end{array} \right\}.$$

reduces to evaluating

$$P_L \triangleq \sum_{T \in S_L} mon(T)$$

Will do dynamic programming to compute $P_L(\,,\,,\,,\ldots,)$

Since, $L \subseteq [k/2]$ & P_L is polytime computable. Done!

Problem 3 :-

$O^*\left(1.657^k\right)$ algorithm for k-path in an undirected graphs.

Idea 1:-

Start with random partition of G.

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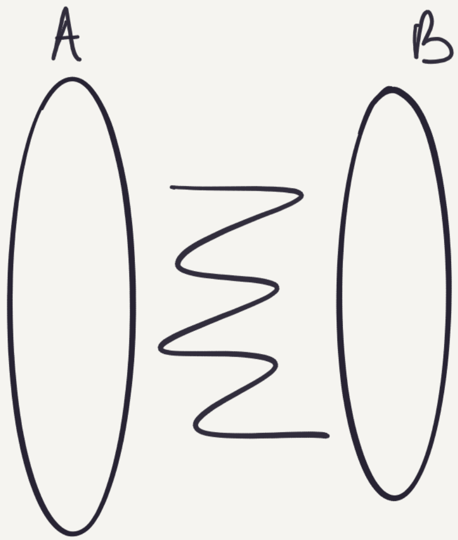
Start with random partition of G.

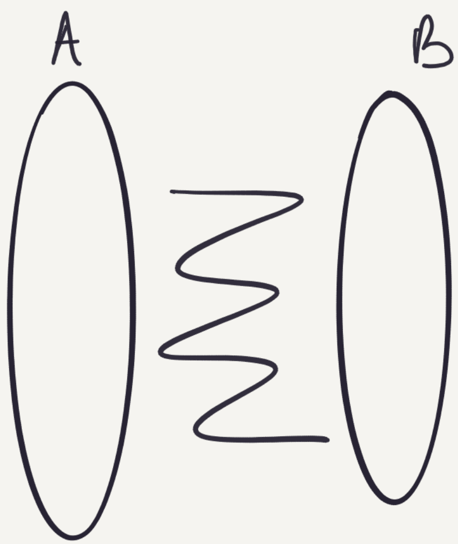


- $A \cap B = \emptyset$
- $A \cup B = V(G)$

Question:- Should be make this partition with choosing a vertex in A with probability $1/2$.

Idea 1:-

Start with random partition of G.



- $A \cap B = \emptyset$
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Question:- Should be make this partition with choosing a vertex in A with probability $1/2$.

This itself will incur 2^k in the running time.

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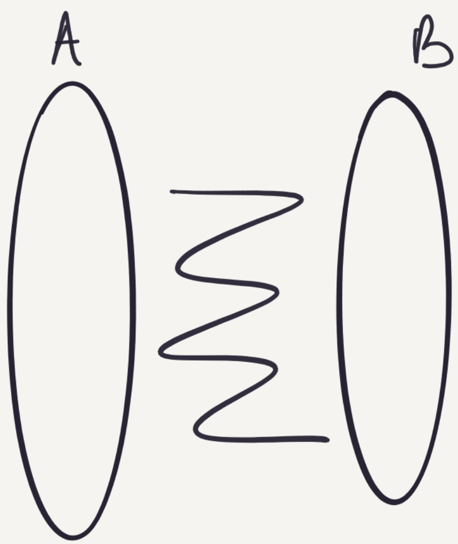
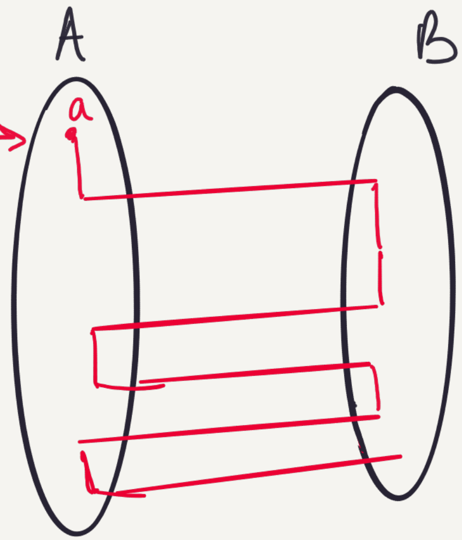


- $A \cap B = \emptyset$
- $A \cup B = V(G)$
- put vertices in A with prob "p".

Idea 2:-

Start with random partition of G.

focus on a path starting at a vertex $a \in A$



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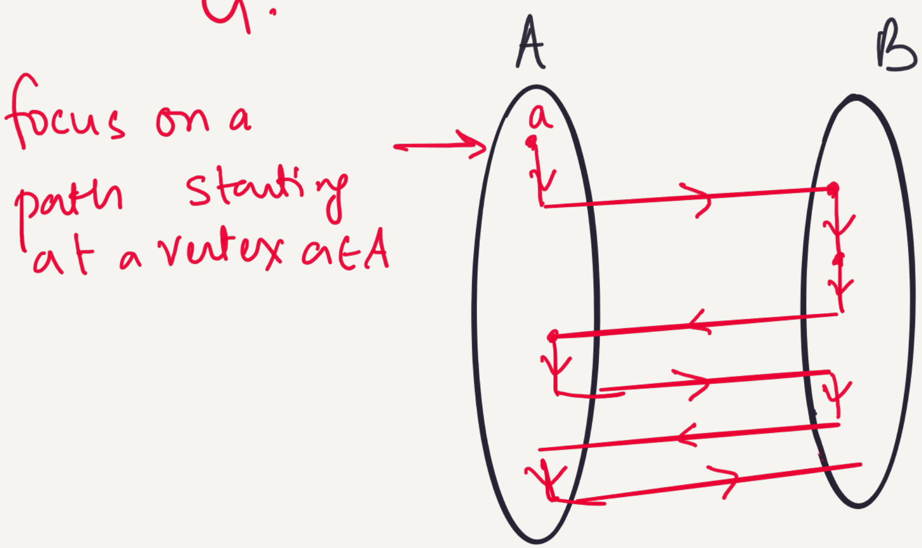

- $A \cap B = \emptyset$
- $A \cup B = V(G)$
- put vertices in A with prob "p".

Variables :- Every vertex in A gets a variable and every edge in B gets a variable!

Idea 2:-

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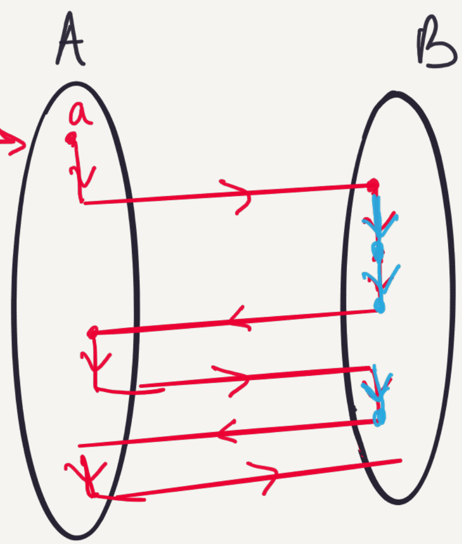

- $A \cap B = \emptyset$
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Variables:- Every vertex in A gets a variable and every edge in B gets a variable!

Savings:- Imagine that every edge variable is associated with its head vertices.

Idea 2:-

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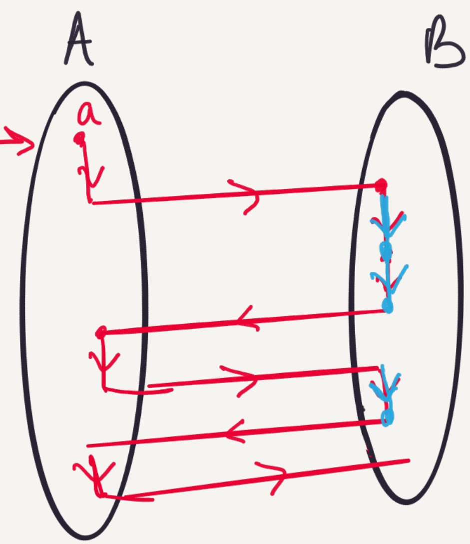


- $A \cap B = \emptyset$
- $A \cup B = V(G)$
- put vertices in A with prob "p".

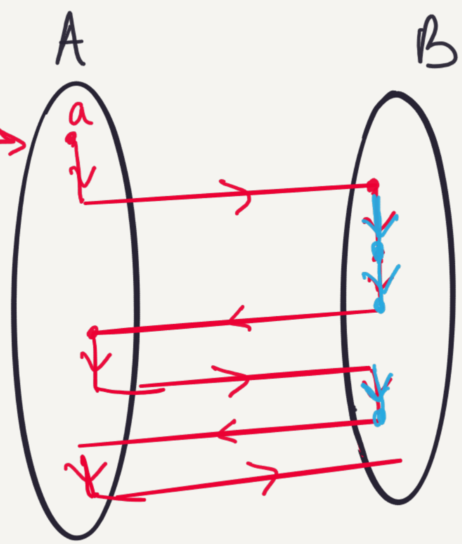
Variables:- Every vertex in A gets a variable and every edge in B gets a variable!

Savings:- Imagine that every edge variable is associated with its head vertices. So every time path goes from A to B, we save a variable.

Idea 2:-

Start with random partition of G.

focus on a path starting at a vertex $a \in A$



- $A \cap B = \emptyset$
- $A \cup B = V(G)$
- put vertices in A with prob "p".

Variables:- : Every vertex in A gets a variable and every edge in B gets a variable!

Algorithm:- $\boxed{2^{k-lq}}$

Here, q = # of times path goes from $A \to B$.

Idea 2:-

Start with random partition of G.

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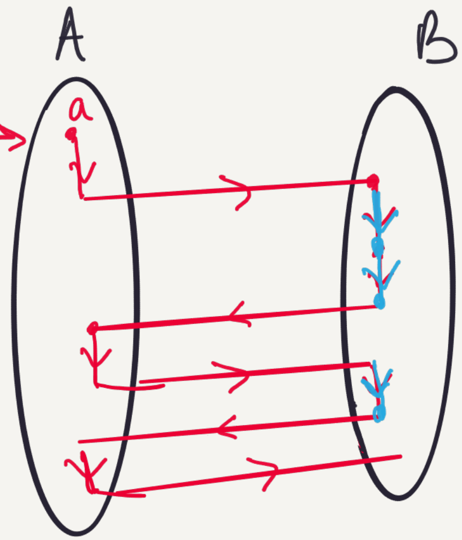

- $A \cap B = \emptyset$
- $A \cup B = V(G)$
- put vertices in A with prob "p".
- We can choose p such that $q = 1/4$ and $\frac{1}{\text{prob}(A,B)} = \text{poly}(n)$.

Variables:- Every vertex in A gets a variable and every edge in B gets a variable!

Algorithm:- $\boxed{2^{k-|q|} \cdot \frac{1}{\text{prob}(A,B)}}$

Here, q = # times path goes from $A \to B$.

That will be all about this.
Any questions.