Advanced Algorithms Homework

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December 08, 2020

Problems

Problem 1 (Truncated SVD as best low-rank approximation)

Let $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ be the truncated SVD. Give a formal proof (check slides for hints) that this gives the best possible rank-k approximation of A, i.e. for any matrix B of rank at most k:

$$\|A - A_k\|_F \leq \|A - B\|_F$$

Problem 2 (Frobenius norm)

1. For any matrix A show that $\sigma_k \leq \frac{\|A\|_F}{\sqrt{k}}$.
2. Prove that there exists a matrix B of rank at most k such that $\|A - B\|_2 \leq \frac{\|A\|_F}{\sqrt{k}}$.
3. Does there exist a matrix B of rank at most k such that $\|A - B\|_F \leq \frac{\|A\|_F}{\sqrt{k}}$? If yes, construct B, if no then give a counterexample.

Problem 3 (Faster power method)

In the lecture we discussed power method: using $(A^T A)^n$ for large enough n to compute the top singular vector. A major drawback of this approach for sparse matrices is that $B = A^T A$ is dense even if A is sparse. Consider an alternative approach: we pick a random Gaussian vector x (each coordinate is i.i.d $\sim N(0, 1)$) and compute $B^n x$. Note that in this case we can compute the resulting expression as $A^T(A(A^T(\ldots A^T(Ax)$, where each each matrix-vector multiplication is sparse and hence can be done in $nnz(A)$ time where nnz is the number of non-zero entries in A.

Show the following statement:

Theorem 0.1. *Let x be a unit vector in $\mathbb{R}^d$ and let v_1 be the top singular vector of A. Suppose that $|x^T v_1| \geq \delta$ and:*

- *V is a subspace spanned by singular vectors v_j such that $\sigma_j \geq (1-\epsilon)\sigma_1$*
- *Let $z = (A^T A)^k x$ for $k = \frac{1}{2\epsilon}\log(1/\epsilon\delta)$ and $w = \frac{z}{\|z\|_2}$ be a unit vector in this direction.*

Let $w = w^{\perp} + w^{\|}$, where $w_{\|}$ lies in V (is a projection on it) and $w^{\perp} \perp V$. Then $\|w^{\perp}\|_2^2 \leq \epsilon$.

How can we use the theorem above to find the top singular vector using the faster power method?