# 5. Lecture notes on matroid intersection

One nice feature about matroids is that a simple greedy algorithm allows to optimize over its independent sets or over its bases. At the same time, this shows the limitation of the use of matroids: for many combinatorial optimization problems, the greedy algorithm does not provide an optimum solution. Yet, as we will show in this chapter, the expressive power of matroids become much greater once we consider the *intersection* of the family of independent sets of *two* matroids.

Consider two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ on the same ground set $E$, and consider the family of indepedent sets common to both matroids, $\mathcal{I}_1 \cap \mathcal{I}_2$. This is what is commonly referred to as the intersection of two matroids.

In this chapter, after giving some examples of matroid intersection, we show that that finding a largest common independent set to 2 matroids can be done efficiently, and provide a min-max relation for the maximum value. We also consider the weighted setting (generalizing the assignment problem), although we will not give an algorithm in the general case (although one exists); we only restrict to a special case, namely the arborescence problem. We shall hint an algorithm for the general case by characterizing the matroid intersection polytope and thereby giving a min-max relation for it (an NP∩co-NP characterization). Finally, we discuss also *matroid union*; a powerful way to construct matroids from other matroids in which matroid intersection plays a central role. (The term 'matroid union' is misleading as it is not what we could expect after having defined matroid intersection... it does *not* correspond to $\mathcal{I}_1 \cup \mathcal{I}_2$.)

## 5.1 Examples

### 5.1.1 Bipartite matchings

Matchings in a bipartite graph $G = (V, E)$ with bipartition $(A, B)$ do not form the independent sets of a matroid. However, they can be viewed as the common independent sets to two matroids; this is the canonical example of matroid intersection.

Let $M_A$ be a partition matroid with ground set $E$ where the partition of $E$ is given by $E = \bigcup\{\delta(v) : v \in A\}$ where $\delta(v)$ denotes the edges incident to $v$. Notice that this is a partition since all edges have precisely one endpoint in $A$. We also define $k_v = 1$ for every $v \in A$. Thus, the family of independent sets of $M_A$ is given by

$$\mathcal{I}_A = \{F : |F \cap \delta(v)| \leq 1 \text{ for all } v \in A\}.$$

In other words, a set of edges is independent for $M_A$ if it has at most one edge incident to every vertex of $A$ (and any number of edges incident to every vertex of $b$). We can similarly define $M_B = (E, \mathcal{I}_B)$ by

$$\mathcal{I}_B = \{F : |F \cap \delta(v)| \leq 1 \text{ for all } v \in B\}.$$

Now observe that any $F \in \mathcal{I}_A \cap \mathcal{I}_B$ corresponds to a matching in $G$, and vice versa. And the largest common independent set to $\mathcal{I}_A$ and $\mathcal{I}_B$ corresponds to a maximum matching in $G$.

### 5.1.2 Arborescences

Given a digraph $D = (V, A)$ and a special root vertex $r \in V$, an $r$-arborescence (or just arborescence) is a spanning tree (when viewed as an undirected graph) directed away from $r$. Thus, in a $r$-arborescence, every vertex is reachable from the root $r$. As an $r$-arborescence has no arc incoming to the root, we assume that $D$ has no such arc.

$r$-arborescences can be viewed as sets simultaneously independent in two matroids. Let $G$ denote the undirected counterpart of $D$ obtained by disregarding the directions of the arcs. Note that if we have both arcs $a_1 = (u, v)$ and $a_2 = (v, u)$ in $D$ then we get two undirected edges also labelled $a_1$ and $a_2$ between $u$ and $v$ in $G$. Define $M_1 = (A, \mathcal{I}_1) = M(G)$ the graphic matroid corresponding to $G$, and $M_2 = (A, \mathcal{I}_2)$ the partition matroid in which independent sets are those with at most one arc incoming to every vertex $v \neq r$. In other words, we let

$$\mathcal{I}_2 = \{F : |F \cap \delta^-(v)| \leq 1 \text{ for all } v \in V \setminus \{r\}\}$$

where $\delta^-(v)$ denotes the set $\{(u, v) \in A\}$ of arcs incoming to $v$. Thus, any $r$-arborescence is independent in both matroids $M_1$ and $M_2$. Conversely, any set $T$ independent in both $M_1$ and $M_2$ and of cardinality $|V| - 1$ (so that it is a base in both matroids) is an $r$-arborescence. Indeed, such a $T$ being a spanning tree in $G$ has a unique path between $r$ and any vertex $v$; this path must be directed from the root $r$ since otherwise we would have either an arc incoming to $r$ or two arcs incoming to the same vertex.

In the minimum cost arborescence problem, we are also given a cost function $c : A \to \mathbb{R}$ and we are interested in finding the minimum cost $r$-arborescence. This is a directed counterpart to the minimum spanning tree problem but, here, the greedy algorithm does not solve the problem.

### 5.1.3 Orientations

Given an undirected graph $G = (V, E)$, we consider orientations of all its edges into directed arcs; namely, each (undirected) edge[1] to emphasize $\{u, v\}$ is either replaced by an arc[2] $(u, v)$ from $u$ to $v$, or by an arc $(v, u)$ from $v$ to $u$. Our goal is, given $k : V \to \mathbb{N}$, to decide whether there exists an orientation such that, for every vertex $v \in V$, the indegree of vertex $v$ (the number of arcs entering $v$) is at most $k(v)$. Clearly, this is not always possible, and this problem can be solved using matroid intersection (or network flows as well).

To attack this problem through matroid intersection, consider the directed graph $D = (V, A)$ in which for every edge $e = \{u, v\}$ of $E$ is replaced by the two arcs $(u, v)$ and $(v, u)$.

[1] Usually, we use $(u, v)$ to denote an (undirected) edge. In this section, however, we use the notation $\{u, v\}$ rather than $(u, v)$ to emphasize that edges are undirected.

[2] We use arcs in the case of directed graphs, and edges for undirected graphs.

With the arc set $A$ as ground set, we define two partition matroids, $M_1$ and $M_2$. To be independent in $M_1$, one can take at most one of $\{(u, v), (v, u)\}$ for every $(u, v) \in E$, i.e.

$$\mathcal{I}_1 = \{F \subseteq A : |F \cap \{(u, v), (v, u)\}| \leq 1 \text{ for all } (u, v) \in E\}.$$

To be independent in $M_2$, one can take at most $k(v)$ arcs among $\delta^-(v)$ for every $v$:

$$\mathcal{I}_2 = \{F \subseteq A : |F \cap \delta^-(v)| \leq k(v) \text{ for all } v \in V\}.$$

Observe that this indeed defines a partition matroid since the sets $\delta^-(v)$ over all $v$ partition $A$.

Therefore, there exists an orientation satisfying the required indegree restrictions if there exists a common independent set to $M_1$ and $M_2$ of cardinality precisely $|E|$ (in which case we select either $(u, v)$ or $(v, u)$ but not both).

### 5.1.4 Colorful Spanning Trees

Suppose we have an undirected graph $G = (V, E)$ and every edge has a color. This is represented by a partition of $E$ into $E_1 \cup \cdots \cup E_k$ where each $E_i$ represents a set of edges of the same color $i$. The problem of deciding whether this graph has a spanning tree in which all edges have a different color can be tackled through matroid intersection. Such a spanning tree is called *colorful*.

Colorful spanning trees are bases of the graphic matroid $M_1 = M(G)$ which are also independent in the partition matroid $M_2 = (E, \mathcal{I}_2)$ defined by $\mathcal{I}_2 = \{F : |F \cap E_i| \leq 1 \text{ for all } i\}$.

### 5.1.5 Union of Two Forests

In Section 5.5, we show that one can decide whether a graph $G$ has two edge-disjoint spanning trees by matroid intersection.

## 5.2 Largest Common Independent Set

As usual, one issue is to find a common independent set of largest cardinality, another is to prove that indeed it is optimal. This is done through a min-max relation.

Given two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ with rank functions $r_1$ and $r_2$ respectively, consider any set $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ and any $U \subseteq E$. Observe that

$$|S| = |S \cap U| + |S \cap (E \setminus U)| \leq r_1(U) + r_2(E \setminus U),$$

since both $S \cap U$ and $S \cap (E \setminus U)$ are independent in $M_1$ and in $M_2$ (by property $(I_1)$); in particular (and this seems weaker), $S \cap U$ is independent for $M_1$ while $S \cap (E \setminus U)$ is independent for $M_2$. Now, we can take the maximum over $S$ and the minimum over $U$ and derive:

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| \leq \min_{U \subseteq E} \left[ r_1(U) + r_2(E \setminus U) \right].$$

Somewhat surprisingly, we will show that we always have equality:

**Theorem 5.1 (Matroid Intersection)** *For any two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ with rank functions $r_1$ and $r_2$ respectively, we have:*

$$\max_{S \in \mathcal{I}_1 \cap \mathcal{I}_2} |S| = \min_{U \subseteq E} \left[ r_1(U) + r_2(E \setminus U) \right].$$

Before describing an algorithm for matroid intersection that proves this theorem, we consider what the min-max result says for some special cases. First, observe that we can always restrict our attention to sets $U$ which are closed for matroid $M_1$. Indeed, if that was not the case, we could replace $U$ by $V = span_{M_1}(U)$ and we would have that $r_1(V) = r_1(U)$ while $r_2(E \setminus V) \leq r_2(E \setminus U)$. This shows that there always exists a set $U$ attaining the minimum which is closed for $M_1$. Similarly, we could assume that $E \setminus U$ is closed for $M_2$ (but both assumptions cannot be made simultaneously).

When specializing the matroid intersection theorem to the graph orientation problem discussed earlier in this chapter, we can derive the following.

**Theorem 5.2** *$G = (V, E)$ has an orientation such that the indegree of vertex $v$ is at most $k(v)$ for every $v \in V$ if and only if for all $P \subseteq V$ we have*[3]*:*

$$|E(P)| \leq \sum_{v \in P} k(v).$$

Similarly, for colorful spanning trees, we obtain:

**Theorem 5.3** *Given a graph $G = (V, E)$ with edges of $E_i$ colored $i$ for $i = 1, \cdots, k$, there exists a colorful spanning tree if and only if deleting the edges of any $c$ colors (for any $c \in \mathbb{N}$) produces at most $c + 1$ connected components.*

The proof of Theorem 5.1 and the corresponding algorithm for finding a maximum cardinality independent set common to two matroids will appear here soon...

**Exercise 5-1.** Deduce König's theorem about the maximum size of a matching in a bipartite graph from the min-max relation for the maximum independent set common to two matroids.

## 5.3 Matroid Intersection Polytope

To be completed.

[3] $E(P)$ denotes the set of edges with both endoints in $P$.

## 5.4 Arborescence Problem

The minimum cost $r$-arborescence is the problem of, given a directed graph $D = (V, A)$, a root vertex $r \in V$ and a cost $c_a$ for every arc $a \in A$, finding an $r$-arborescence in $D$ of minimum total cost. This can thus be viewed as a weighted matroid intersection problem and we could use the full machinery of matroid intersection algorithms and results. However, here, we are going to develop a simpler algorithm using notions similar to the Hungarian method for the assignment problem. We will assume that the costs are nonnegative.

As an integer program, the problem can be formulated as follows. Letting $x_a$ be 1 for the arcs of an $r$-arborescence, we have the formulation:

$$
\begin{aligned}
OPT \quad = \quad \min \quad & \sum_{a \in A} c_a x_a \\
\text{subject to:} \quad & \\
& \sum_{a \in \delta^-(S)} x_a \geq 1 && \forall S \subseteq V \setminus \{r\} \\
& \sum_{a \in \delta^-(v)} x_a = 1 && \forall v \in V \setminus \{r\} \\
& x_a \in \{0, 1\} && a \in A.
\end{aligned}
$$

In this formulation $\delta^-(S)$ represents the set of arcs $\{(u, v) \in A : u \notin S, v \in S\}$. One can check that any feasible solution to the above corresponds to the incidence vector of an $r$-arborescence. Notice that this optimization problem has an exponential number of constraints. We are going to show that we can relax both the integrality restrictions to $x_a \geq 0$ and also remove the equality constraints $\sum_{a \in \delta^-(v)} x_a = 1$ and still there will be an $r$-arboresence that will be optimum for this relaxed (now linear) program. The relaxed linear program (still with an exponential number of constraints) is:

$$
\begin{aligned}
LP \quad = \quad \min \quad & \sum_{a \in A} c_a x_a \\
\text{subject to:} \quad & \\
(P) \qquad & \sum_{a \in \delta^-(S)} x_a \geq 1 && \forall S \subseteq V \setminus \{r\} \\
& x_a \geq 0 && a \in A.
\end{aligned}
$$

The dual of this linear program is:

$$(D)\quad \begin{array}{rcll} LP & = & \max & \sum_{S\subseteq V\setminus\{r\}} y_S \\ & & \text{subject to:} & \\ & & & \sum_{S:a\in\delta^-(S)} y_S \le c_a \\ & & & y_S \ge 0 \qquad\qquad S\subseteq V\setminus\{r\}. \end{array}$$

The algorithm will be constructing an arborescence $T$ (and the corresponding incidence vector $x$ with $x_a = 1$ whenever $a \in T$ and 0 otherwise) and a feasible dual solution $y$ which satisfy complementary slackness, and this will show that $T$ corresponds to an optimum solution of $(P)$, and hence is an optimum arborescence. Complementary slackness says:

1. $y_S > 0 \Longrightarrow |T \cap \delta^-(S)| = 1$, and
2. $a \in T \Longrightarrow \sum_{S:a\in\delta^-(S)} y_S = c_a$.

The algorithm will proceed in 2 phases. In the first phase, it will construct a dual feasible solution $y$ and a set $F$ of arcs which has a directed path from the root to every vertex. This may not be an $r$-arborescence as there might be too many arcs. The arcs in $F$ will satisfy condition 2 above (but not condition 1). In the second phase, the algorithm will remove unnecessary arcs, and will get an $r$-arborescence satisfying condition 1.

Phase 1 is initialized with $F = \emptyset$ and $y_S = 0$ for all $S$. While $F$ does not contain a directed path to every vertex in $V$, the algorithm selects a set $S$ such that (i) inside $S$, $F$ is strongly connected (i.e. every vertex can reach every vertex) and (ii) $F\cap\delta^-(S) = \emptyset$. This set $S$ exists since we can contract all strongly connected components and in the resulting acyclic digraph, there must be a vertex (which may be coming from the shrinking of a strongly connected component) with no incoming arc (otherwise tracing back from that vertex we would either get to the root or discover a new directed cycle (which we could shrink)). Now we increase $y_S$ as much as possible until a new inequality, say for arc $a_k$, $\sum_{S:a_k\in\delta^-(S)} y_S \le c_{a_k}$ becomes an equality. In so doing, the solution $y$ remains dual feasible and still satisfies condition 2. We can now add $a_k$ to $F$ without violating complementary slackness condition 2, and then we increment $k$ (which at the start we initialized at $k = 1$). And we continue by selecting another set $S$, and so on, until every vertex is reachable from $r$ in $F$. We have now such a set $F = \{a_1, a_2, \cdots, a_k\}$ and a dual feasible solution $y$ satisfying condition 2.

In step 2, we eliminate as many arcs as possible, but we consider them in *reverse order* they were added to $F$. Thus, we let $i$ go from $k$ to 1, and if $F \setminus \{a_i\}$ still contains a directed path from $r$ to every vertex, we remove $a_i$ from $F$, and continue. We then output the resulting set $T$ of arcs.

The first claim is that $T$ is an arborescence. Indeed, we claim it has exactly $|V| - 1$ arcs with precisely one arc incoming to every vertex $v \in V \setminus \{r\}$. Indeed, if not, there would be two arcs $a_i$ and $a_j$ incoming to some vertex $v$; say that $i < j$. In the reverse delete step, we should have removed $a_j$; indeed any vertex reachable from $r$ through $a_j$ could be reached through $a_i$ as well (unless $a_i$ is unnecessary in which case we could get rid of $a_i$ later on).

The second (and final) claim is that the complementary slackness condition 1 is also satisfied. Indeed, assume not, and assume that we have a set $S$ with $y_S > 0$ and $|T \cap \delta^-(S)| > 1$. $S$ was chosen at some point by the algorithm and at that time we added $a_k \in \delta^-(S)$ to $F$. As there were no other arcs in $\delta^-(S)$ prior to adding $a_k$ to $F$, it means that all other arcs in $T \cap \delta^-(S)$ must be of the form $a_j$ with $j > k$. In addition, when $S$ was chosen, $F$ was already strongly connected within $S$; this means that from any vertex inside $S$, one can go to any other vertex inside $S$ using arcs $a_i$ with $i < k$. We claim that when $a_j$ was considered for removal, it should have been removed. Indeed, assume that $a_j$ is needed to go to vertex $v$, and that along the path $P$ to $v$ the last vertex in $S$ is $w \in S$. Then we could go to $v$ by using $a_k$ which leads somewhere in $S$ then take arcs $a_i$ with $i < k$ (none of which have been removed yet as $i < k < j$) to $w \in S$ and then continue along path $P$. So $a_j$ was not really necessary and should have been removed. This shows that complementary slackness condition 1 is also satisfied and hence the arborescence built is optimal.

## 5.5 Matroid Union

From any matroid $M = (E, \mathcal{I})$, one can construct a dual matroid $M^* = (E, \mathcal{I}*)$.

**Theorem 5.4** *Let $\mathcal{I}* = \{X \subseteq E : E \setminus X$ contains a base of $M\}$. Then $M^* = (E, \mathcal{I}*)$ is a matroid with rank function*

$$r_{M^*}(X) = |X| + r_M(E \setminus X) - r_M(E).$$

There are several ways to show this. One is to first show that indeed the size of the largest subset of $X$ in $\mathcal{I}*$ has cardinality $|X| + r_M(E \setminus X) - r_M(E)$ and then show that $r_{M^*}$ satisfies the three conditions that a rank function of a matroid needs to satisfy (the third one, submodularity, follows from the submodularity of the rank function for $M$).

One can use Theorem 5.4 and matroid intersection to get a good characterization of when a graph $G = (V, E)$ has two edge-disjoint spanning trees. Indeed, letting $M$ be the graphic matroid of the graph $G$, we get that $G$ has two edge-disjoint spanning trees if and only if

$$\max_{S \in \mathcal{I} \cap \mathcal{I}*} |S| = |V| - 1.$$

For the graphic matroid, we know that $r_M(F) = n - \kappa(F)$ where $n = |V|$ and $\kappa(F)$ denotes the number of connected components of $(V, F)$. But by the matroid intersection theorem, we can write:

$$\begin{aligned}
\max_{S \in \mathcal{I} \cap \mathcal{I}*} |S| &= \min_{E_1 \subseteq E} \left[ r_M(E_1) + r_{M^*}(E \setminus E_1) \right] \\
&= \min_{E_1 \subseteq E} \left[ (n - \kappa(E_1)) + (|E \setminus E_1| + \kappa(E) - \kappa(E_1)) \right] \\
&= \min_{E_1 \subseteq E} \left[ n + 1 + |E \setminus E_1| - 2\kappa(E_1) \right],
\end{aligned}$$

where we replaced $\kappa(E)$ by 1 since otherwise $G$ would even have one spanning tree. Rearranging terms, we get that $G$ has two edge-dsjoint spanning trees if and only if for all $E_1 \subseteq E$, we have that $E \setminus E_1 \geq 2(\kappa(E_1) - 1)$. If this inequality is violated for some $E_1$, we can add to $E_1$ any edge that does not decrease $\kappa(E_1)$. In other words, if the connected components of $E_1$ are $V_1, V_2, \cdots, V_p$ then we can assume that $E_1 = E \setminus \delta(V_1, V_2, \cdots V_p)$ where $\delta(V_1, \cdots, V_p) = \{(u,v) \in E : u \in V_i, v \in V_j \text{ and } i \neq j\}$. Thus we have shown:

**Theorem 5.5** *$G$ has two edge-disjoint spanning trees if and only if for all partitions $V_1$, $V_2, \cdots V_p$ of $V$, we have*

$$|\delta(V_1, \cdots, V_p)| \geq 2(p-1).$$

Theorem 5.5 can be generalized to an arbitrary number of edge-disjoint spanning trees. This result is not proved here.

**Theorem 5.6** *$G$ has $k$ edge-disjoint spanning trees if and only if for all partitions $V_1$, $V_2, \cdots V_p$ of $V$, we have*

$$|\delta(V_1, \cdots, V_p)| \geq k(p-1).$$

From two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, we can also define its union by $M_1 \cup M_2 = (E, \mathcal{I})$ where $\mathcal{I} = \{S_1 \cup S_2 : S_1 \in \mathcal{I}_1, S_2 \in \mathcal{I}_2\}$. Notice that we do not impose the two matroids to be identical as we just did for edge-disjoint spanning trees.

We can show that:

**Theorem 5.7** *$M_1 \cup M_2$ is a matroid. Furthermore its rank function is given by*

$$r_{M_1 \cup M_2}(S) = \min_{F \subseteq S} \{|S \setminus F| + r_{M_1}(F) + r_{M_2}(F)\}.$$

**Proof:** To show that it is a matroid, assume that $X, Y \in \mathcal{I}$ with $|X| < |Y|$. Let $X = X_1 \cup X_2$ and $Y = Y_1 \cup Y_2$ where $X_1, Y_1 \in \mathcal{I}_1$ and $X_2, Y_2 \in \mathcal{I}_2$. We can furthermore assume that the $X_i$'s are disjoint and so are the $Y_i$'s. Finally we assume that among all choices for $X_1, X_2, Y_1$ and $Y_2$, we choose the one maximizing $|X_1 \cap Y_1| + |X_2 \cap Y_2|$. Since $|Y| > |X|$, we can assume that $|Y_1| > |X_1|$. Thus, there exists $e \in (Y_1 \setminus X_1)$ such that $X_1 \cup \{e\}$ is independent for $M_1$. The maximality implies that $e \notin X_2$ (otherwise consider $X_1 \cup \{e\}$ and $X_2 \setminus \{e\}$). But this implies that $X \cup \{e\} \in \mathcal{I}$ as desired.

We now show the expression for the rank function. The fact that it is $\leq$ is obvious as an independent set $S \in \mathcal{I}$ has size $|S \setminus F| + |S \cap F| \leq |S \setminus F| + r_{M_1}(F) + r_{M_2}(F)$ and this is true for any $F$.

For the converse, let us prove it for the entire ground set $S = E$. Once we prove that

$$r_{M_1 \cup M_2}(E) = \min_{F \subseteq S} \{|E \setminus F| + r_{M_1}(F) + r_{M_2}(F)\},$$

the corresponding statement for any set $S$ will follow by just restricting our matroids to $S$.

Let $X$ be a base of $M_1 \cup M_2$. The fact that $X \in \mathcal{I}$ means that $X = X_1 \cup X_2$ with $X_1 \in \mathcal{I}_1$ and $X_2 \in \mathcal{I}_2$. We can furthermore assume that $X_1$ and $X_2$ are disjoint and that

$r_{M_2}(X_2) = r_{M_2}(E)$ (otherwise add elements to $X_2$ and possibly remove them from $X_1$). Thus we can assume that $|X| = |X_1| + r_{M_2}(E)$. We have that $X_1 \in \mathcal{I}_1$ and also that $X_1$ is independent for the dual of $M_2$ (as the complement of $X_1$ contains a base of $M_2$). In other words, $X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*$. The proof is completed by using the matroid intersection theorem and Theorem 5.4:

$$\begin{aligned} r_{M_1 \cup M_2}(E) = |X| &= \max_{X_1 \in \mathcal{I}_1 \cap \mathcal{I}_2^*} (|X_1| + r_{M_2}(E)) \\ &= \min_{E_1 \subseteq E} \left( r_{M_1}(E_1) + r_{M_2^*}(E \setminus E_1) + r_{M_2}(E) \right) \\ &= \min_{E_1 \subseteq E} (r_{M_1}(E_1) + |E \setminus E_1| + r_{M_2}(E_1) - r_{M_2}(E) + r_{M_2}(E)) \\ &= \min_{E_1 \subseteq E} (|E \setminus E_1| + r_{M_1}(E_1) + r_{M_2}(E_1)), \end{aligned}$$

as desired. △

Since Theorem 5.7 says that $M_1 \cup M_2$ is a matroid, we know that its rank function is submodular. This is, however, not obvious from the formula given in the theorem.

### 5.5.1 Spanning Tree Game

The spanning tree game is a 2-player game. Each player in turn selects an edge. Player 1 starts by deleting an edge, and then player 2 fixes an edge (which has not been deleted yet); an edge fixed cannot be deleted later on by the other player. Player 2 wins if he succeeds in constructing a spanning tree of the graph; otherwise, player 1 wins. The question is which graphs admit a *winning strategy* for player 1 (no matter what the other player does),a nd which admit a winning strategy for player 2.

**Theorem 5.8** *For the spanning tree game on a graph $G = (V, E)$, player 1 has a winning strategy if and only if $G$ does not have two edge-disjoint spanning trees. Otherwise, player 2 has a winning strategy.*

If $G$ does not have 2 edge-disjoint spanning trees then, by Theorem 5.5, we know that there exists a partition $V_1, \cdots, V_p$ of $V$ with $|\delta(V_1, \cdots, V_p)| \le 2(p-1) - 1$. The winning strategy for player 1 is then to always delete an edge from $\delta(V_1, \cdots, V_p)$. As player 1 plays before player 2, the edges in $\delta(V_1, \cdots, V_p)$ will be exhausted before player 2 can fix $p-1$ of them, and therefore player 2 loses. The converse is the subject of exercise 5-4.

**Exercise 5-2.** Derive from theorem 5.7 that the union of $k$ matroids $M_1$, $M_2, \cdots, M_k$ is a matroid with rank function

$$r_{M_1 \cup M_2 \cup \cdots \cup M_k}(S) = \min_{F \subseteq S} \{|S \setminus F| + r_{M_1}(F) + r_{M_2}(F) + \cdots + r_{M_k}(F)\}.$$

**Exercise 5-3.** Derive Theorem 5.6 from Exercise 5-2.

**Exercise 5-4.** Assume that $G$ has 2 edge-disjoint spanning trees. Give a winning strategy for player 2 in the spanning tree game.

**Exercise 5-5.** Find two edge-disjoint spanning trees in the following graph with 16 vertices and 30 edges or prove that no such trees exist.