EECS 495: Combinatorial Optimization **Lecture 11**
Submodular Functions

Reading: Schrijver, Chapter 44

Announcements

- Homeworks: only required to do 10 total (so 5 more). They are due by March 9 (last day of quarter).
- Lectures: there is no lecture next Wed. Instead, schedule a two-hour meeting with me to present your reading project on Wed. or Thu. (time slots 10-12, 12-2, 2-4 on Wed., 10-12, 3-5 on Thu.)
  - Presentations are 1.5 hours long.
  - Be prepared to explain how you will divide the work.
  - Bring notes indicating exactly what you plan to write on the board.
  - Present your project to me. This presentation will factor into your final presentation grade.
- Reading Projects:
  - schedule a presentation on Feb. 28, March 2, or March 7.
  - prepare a handout at least one day before your presentation with sections: introduction (informal problem statement, motivation), model (formal problem statement), solution (list techniques you're using; if we haven't discussed them yet, define them), followup questions (at least two, these can be "exercises" to help us understand things, or "open questions"; at least one should be an exercise)

Submodularity

Def: $f : 2^S \to \mathbb{R}$ is *submodular* if

$\forall A, B \subseteq S, f(A)+f(B) \geq f(A\cap B)+f(A\cup B)$

or equivalently,

$$\forall A \subseteq B, e \notin B,$$

$$f(A+e) - f(A) \geq f(B+e) - f(B).$$

Properties

- *non-negative* if $f(A) \geq 0$
- *symmetric* if $f(A) = f(S \setminus A)$
- *monotone (non-decreasing)* if $f(A) \leq f(B) \forall A \subseteq B$
- *integer-valued* if $f(A) \in \mathbb{Z}$

Example: Matroids:

Rank function, sum of rank functions $r_1(U) + r_2(S \setminus U)$.

Example: Cuts:

Given $G = (V, E)$ and capacity $c : E \to \mathbb{R}^+$, *cut function* $f : 2^V \to \mathbb{R}^+$ is $f(U) = \sum_{e\in\delta(U)} c(e)$.

Example: Coverage:

Let $\{T_1, \ldots, T_n\}$ be subsets of T and $S = \{1, \ldots, n\}$. The *coverage function* $f : 2^S \to \mathbb{R}^+$ is $f(A) = |\cup_{i \in A} T_i|$

$\left[\left[\right.\right.$*More generally,* $f(A) = g(\cup_{i\in A} T_i)$ *for monotone submod func* g*.*$\left.\left.\right]\right]$

Example: Flows:

Le t$D = (V, A)$ be directed graph with arc-capacity $c : A \to \mathbb{R}^+$ and $t \in V$ the *sink*. For $U \subseteq V \setminus \{t\}$, $f(U) =$ max flow from U to t in D from sources U.

Minimization

Question: Find $U \subset S$ that minimizes $f(U)$.

Note: Assume wlog $f(\emptyset) = 0$ (add constant if necessary).

[[*f may be negative.*]]

Example: Matroid intersection is submodular function minimization.

Idea: (obvious): evaluation f on all possible subsets.

- exponential time
- no better way for general set functions
- for submodular ones, use structure to get better alg

Goal: Alg that minimizes submod f in polytime given oracle access to function.

Polymatroids

Question: certificate of optimality

For set function f on ground set S with $f(\emptyset) = 0$, define polyedron:

$$P_f = \{x \in \mathbb{R}^S | \sum_{e \in U} x_e \leq f(U) \forall U \subseteq S\}.$$

Example: Draw P_f for $S = \{1, 2\}$, $f(\emptyset) = f(\{1, 2\}) = 0$, $f(\{1\}) = 1$, $f(\{2\}) = -1$.

Def: A polyhedron P is a *polymatroid* if there's a submod f such that $P = P_f$.

Def: Vector $x \in P_f$ is a *base vector* of P_f or f if $x(S) = f(S)$. The *base polytope* $B_f = \{$ base vectors of $f\}$ and is a face of P_f:

$$B_f = \{x \in \mathbb{R}^S | x(U) \leq f(U), x(S) = f(S)\}.$$

Example: Show base polytope of previous example.

Note: B_f bounded since $f(\{s\}) \geq x_s = x(S) - x(S \setminus \{s\}) \geq f(S) - f(S \setminus \{s\})$.

Idea: Min-max theorem that certifies optimality in terms of base polytope:

Claim: Let $f : 2^S \to \mathbb{Z}$ be a submodular function such that $f(\emptyset) = 0$. Then

$$\min_{U \subseteq S} f(U) = \max\{x^-(S) | x \in B_f\}$$

where $x^-(U) = \sum_{i \in U} \min(0, x_i)$.

Note: max $\leq$ min: $x^-(S) \leq x(U) \leq f(U)$ for any subset $U \subseteq S$ and base $x \in B_f$.

Proof of hard direction follows.

Claim: Moreover, given maximizer x, $U = \{i \in S : x_i < 0\}$ is minimizer.

Idea: To minimize submod func, max concave func over base polydron.

- want max sum of neg elts
- sum of all elts const

- thus want min sum pos elts
- intuitively means minimize norm

Claim: For x^* minimizer of

$$\min_x ||x||_2^2 \ s.t.\ x \in B_f$$

a minimizer U^* for f is

$$U^* = \{i \in S : x_i^* \leq 0\}$$

Problem: To run ellipsoid, must test $x \in B_f$.

Note: optimization implies separation (polar polytopes).

Proof of Min-Max

Claim: For $x \in P_f$, define $\mathcal{F}_x = \{U \subseteq S | x(U) = f(U)\}$ (tight constraints). Then $\mathcal{F}_x$ is closed under taking unions and intersections.

Proof: For any $U, V \in \mathcal{F}_x$, have

$$\begin{aligned} f(U \cup V) &\geq x(U \cup V) \\ &= x(U) + x(V) - x(U \cap V) \\ &\geq f(U) + f(V) - f(U \cap v) \\ &\geq f(U \cup V). \end{aligned}$$

Proof: (of min-max relation): Let x be maximizer. Note for any $i, j \in S$ with and $x_i < 0, x_j > 0$, $\exists U_{ij}$ s.t.

1. $i \in U_{ij}, j \notin U_{ij}$
2. $x(U_{ij}) = f(U_{ij})$

since

- Suppose not, i.e. $x(U) < f(U)$ for all such U
- Let $\epsilon' = \min\{f(U) - x(U) | i \in U, j \notin U\}$.
- Let $\epsilon = \min\{\epsilon', |x_i|, |x_j|\}$.
- Define $\hat{x}_i = x_i + \epsilon, \hat{x}_j = x_j - \epsilon, \hat{x}_k = x_k$ for all other $k \in S$.
- Then $\hat{x} \in B_f$ and $\hat{x}^-(S) = x^-(S) + \epsilon > x^-(S)$ contradicting x is maximizer.

Define V to contain neg elts of S:

$$V = \cup_{i:x_i<0} \cap_{j:x_j>0} U_{ij}.$$

Then

- lemma implies $x(V) = f(V)$
- since V contains all neg elts, $x^-(S) = x(V)$
- thus $x^-(S) = f(V)$.

So min-max relation satisfied by set claimed in lemma.

Note: For norm claim, note $||x_\epsilon||_2^2 < ||x||_2^2$.

Optimize over Polymatroid

Idea: Extend greedy alg for matroids.

Problem:

$$\max w^T x \ s.t.\ x \in B_f$$

for w with $w_1 \geq \ldots \geq w_n \geq 0$.

Idea: Greedy

1. set x_1 as high as possible to get large w_1 value: $x_1 = f(\{v_1\})$
2. subject to this, set x_2 as high as possible:
   - $x_2 \leq f(\{v_2\})$
   - $x_1 + x_2 \leq f(\{v_1, v_2\})$

- by submod, $f(\{v_1, v_2\}) - f(\{v_1\}) \leq f(\{v_2\})$

set $x_2 = f(\{v_1, v_2\}) - f(\{v_1\})$.

3. ...

Algorithm: Given total ordering $\prec$ of S, let

- $v_1 \prec \ldots \prec v_n$
- $V_k^{\prec} = \{v_1, \ldots, v_k\}$

Set $b_1^{\prec} \rightarrow f(\{v_1\})$
For $k = 2 \ldots n$, set $b_k^{\prec} \rightarrow f(V_k^{\prec}) - f(V_{k-1}^{\prec})$

Note: need only ordering, actual weights irrelevant.

Claim: Gready outputs optimal feasible vector.
[[*feasibility by induction, optimality by duality as for matroid greedy alg, exercise.*]]

Note: outputs extreme points of base polytope, but want

$$max_{x \in B_f} x^-(V)$$

which may not be extreme point.

[[*Hammer: optimization implies separation, use ellipsoid. More intuition:*]]

Idea: Search over convex combinations of extreme points:

- maintain $x = \sum_k \lambda_k b^{\prec_k}$ where $\lambda_k \geq 0, \sum_k \lambda_k = 1$
- iteratively modify bases in sum by swapping elts in $\prec$, and update weights