Synchronizing Finite Automata

VI-VII. Aperiodic Automata

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Deterministic finite automata (DFA): $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$.

- Q the state set
- Σ the input alphabet
- $\delta : Q \times \Sigma \to Q$ the transition function

$\mathscr{A}$ is called synchronizing if there exists a word $w \in \Sigma^*$ whose action resets $\mathscr{A}$, that is, leaves the automaton in one particular state no matter which state in Q it started at: $\delta(q, w) = \delta(q', w)$ for all $q, q' \in Q$.

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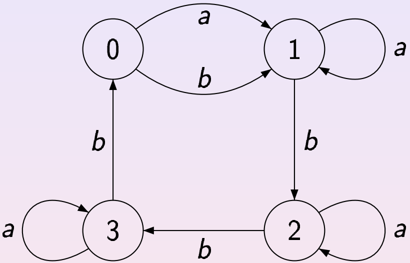

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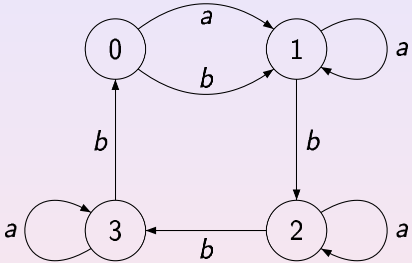

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3. Černý Conjecture

The Černý conjecture is the claim that every synchronizing automaton with n states possesses a reset word of length $(n-1)^2$. The validity of the conjecture is main open problem of the area.

Define the *Černý function* $C(n)$ as the maximum length of shortest reset words for synchronizing automata with n states. In terms of this function, our current knowledge can be summarized in one line:

The Černý conjecture thus claims that in fact $C(n) = (n-1)^2$.

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4. Restriction to Subclasses

Since the Černý problem has proved to be hard in general, a natural approach is to consider its restrictions to various special classes of automata.

We have already mentioned several classes in which synchronizing automata have been investigated with (at least partial) success:

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Recall that the transition monoid of a DFA $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ consists of all transformations $\delta(\sqcup, w) : Q \to Q$ induced by words $w \in \Sigma^*$.

A monoid is said to be aperiodic if all its subgroups are singletons.

A DFA is called aperiodic (or counter-free) if its transition monoid is aperiodic.

An equivalent 'elementary' formulation: $\mathscr{A} = \langle Q, \Sigma, \delta\rangle$ is aperiodic iff for every $q \in Q$ and every $w \in \Sigma^*$ there exists a positive integer m such that $q\,.\,w^m = q\,.\,w^{m+1}$.

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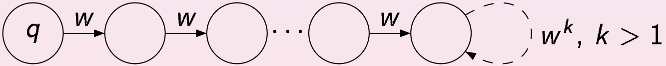

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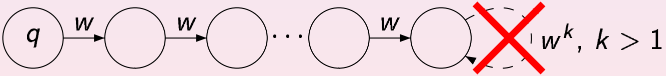

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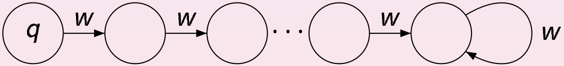
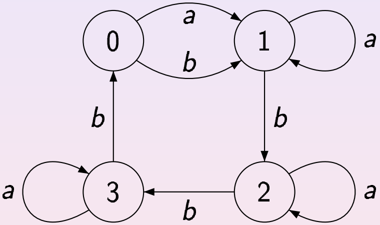


The Černý automaton $\mathscr{C}_4$ is not aperiodic since the letter b acts as a cyclic permutation of the states and thus generates a 4-element subgroup in the transition monoid of $\mathscr{C}_4$.



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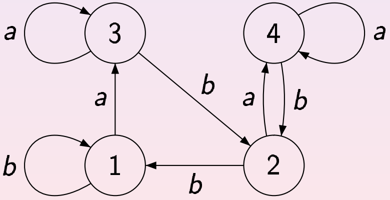

6. Examples

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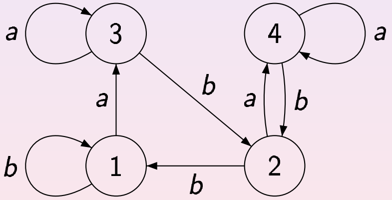

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a	3	4	3	4
b	1	1	2	2

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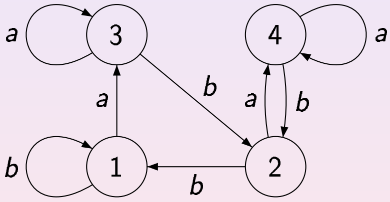

	1	2	3	4
a	3	4	3	4
b	1	1	2	2
ab	2	2	2	2
ba	3	3	4	4
b^2	1	1	1	1

$a^2 = a$

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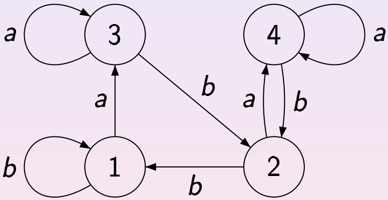

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aba	4	4	4	4
b^2a	3	3	3	3

$a^2 = a,\ ab^2 = b^2,\ bab = ab,\ b^3 = b^2$

In general, there is no way to verify whether or not a given DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is aperiodic avoiding the calculation of its transition monoid and the cardinality of the monoid can reach $|Q|^{|Q|}$. The problem is known to be PSPACE-complete (Sang Cho and Dung T. Huynh, "Finite-automaton aperiodicity is PSPACE-complete", Theor. Comput. Sci. 88 (1991) 99–116).

Also, it is easy to see that synchronization issues remain difficult when restricted to the class of aperiodic automata. Indeed, inspecting the reduction from SAT to SHORT-RESET-WORD shown in Lecture 2, one can see that the construction results in an aperiodic automaton, and therefore, the question of whether or not a given aperiodic automaton admits a reset word whose length does not exceed a given positive integer, is NP-complete.

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8. Monotonicity

In some cases, however, aperiodicity is granted.

A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is monotonic if Q admits a linear order $\leq$ such that, for each $a \in \Sigma$, the transformation $\delta(\sqcup, a)$ preserves $\leq$:

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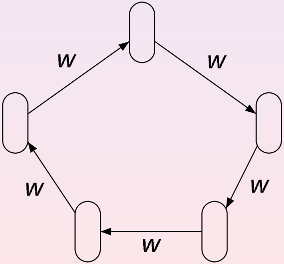

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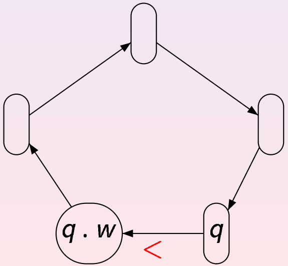

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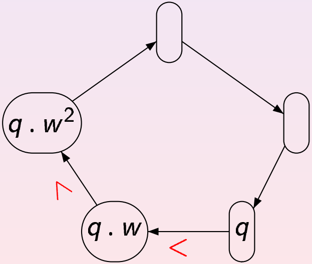

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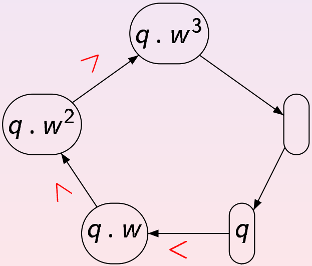

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A DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ is monotonic if Q admits a linear order $\leq$ such that, for each $a \in \Sigma$, the transformation $\delta(\sqcup, a)$ preserves $\leq$:

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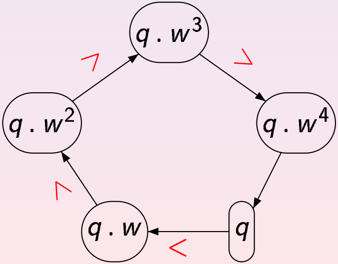

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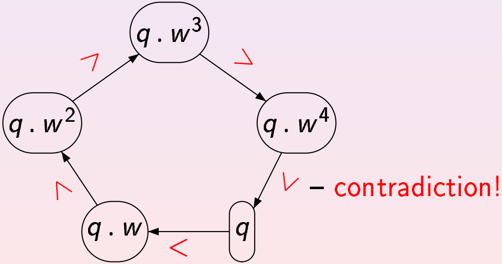

9. Generalized Monotonicity

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We call a DFA $\mathscr{A}$ generalized monotonic of level ℓ if it admits a strictly increasing chain of stable binary relations $\rho_0 \subset \rho_1 \subset \cdots \subset \rho_\ell$, satisfying the following conditions:

- ρ_0 is the equality;
- for each $i = 1, \ldots, \ell$, the congruence π_{i-1} generated by ρ_{i-1} is contained in ρ_i and the relation ρ_i/π_{i-1} is a linear order on each π_i/π_{i-1}-class;
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Monotonic automata are precisely generalized monotonic automata of level 1.

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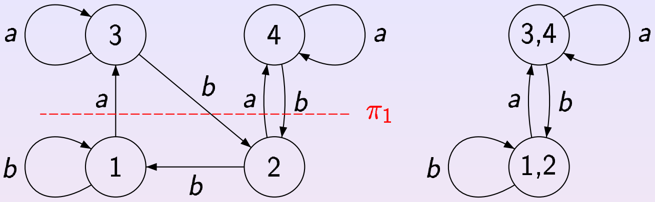

Endowing Q with the order $\leq_1$ such that $1 <_1 2$ and $3 <_1 4$, we get a linear order on each π_1-class. If we order Q/π_1 by letting $\{1,2\} <_2 \{3,4\}$, the quotient automaton becomes monotonic.
It can be shown that the automaton is not monotonic. Moreover, it cannot be emulated by any monotonic automaton.

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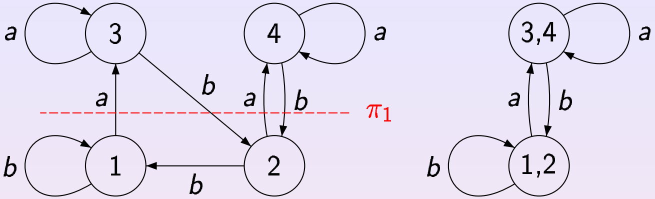

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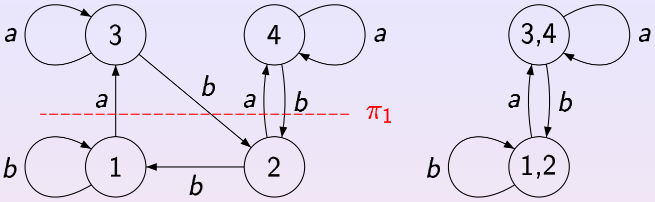

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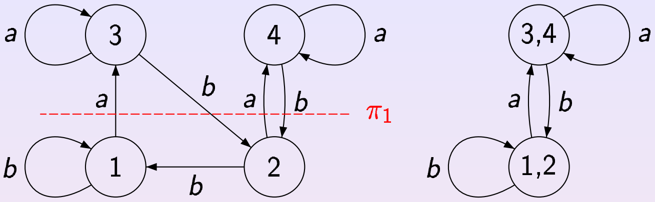

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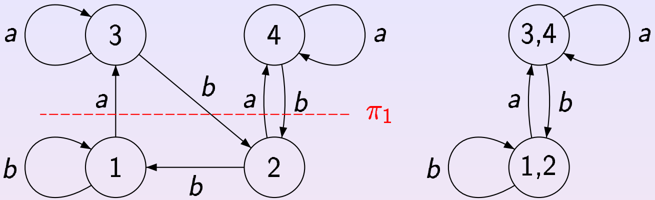

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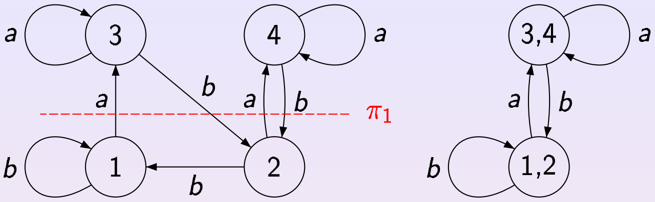

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The Kleene star is clearly the most 'infinite' operation. One cannot eliminate it because neither union nor product can produce infinite languages from finite ones. However, one can use also complement (the class of regular languages is closed under complement by Kleene's theorem). An extended regular expression is built from words by using union, product, Kleene star, and complement, say, $((a+ba)^c ab)^*(b+(aa)^c)^*$.

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Here some abbreviations are used to make the formula readable. Clearly, 0 stands for the empty set and it can be expressed as $(1+1^c)^c$ while $E_1 \setminus E_2 = E_1 \cap E_2^c$ can be expressed as $(E_1^c + E_2)^c$ by De Morgan's law.

However, for the language $(a^2)^*$ that looks alike $(ab)^*$ we would not be able to construct a star-free extended regular expression.

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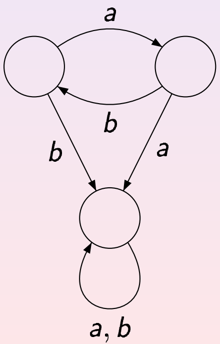
For instance, for $(ab)^*$ and $(a^2)^*$ the minimal automata are



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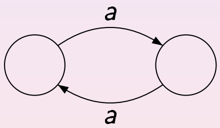

15. Extended Star Height Problem

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It is known that there exist regular languages of any given star height and that, given a language, its star height can be decided. However analogous problems are open for extended star height.

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Extended Star Height Problem

Is there a regular language of extended star height > 1?

Is the class of languages of extended star height 1 decidable?

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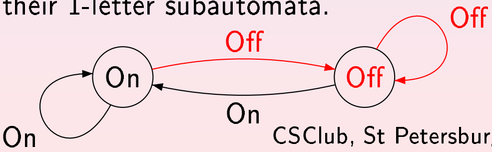

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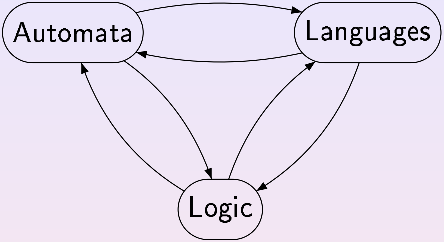
Group Complexity Problem

Given a finite automaton $\mathscr{A}$, can one decide the group complexity of $\mathscr{A}$?
In particular, can we decide if the group complexity of $\mathscr{A}$ is 1?

There is a magic triangle



Logic for words has first order variables (positions) that take values in $\{1, 2, \dots\}$, second order variables (sets of positions) whose values are subsets of $\{1, 2, \dots\}$, the usual connectives and quantifiers, the predicate symbol $<$ with the usual meaning (and maybe some additional numerical predicates), and a special predicate Q_a for each letter a with the meaning: $Q_a x$ is true iff the position x holds the letter a.

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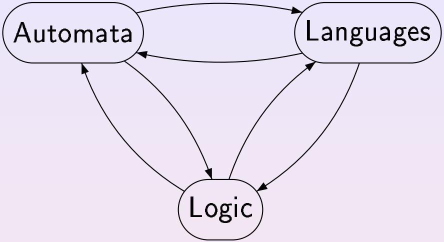

Logic for words has first order variables (positions) that take values in $\{1, 2, \dots\}$, second order variables (sets of positions) whose values are subsets of $\{1, 2, \dots\}$, the usual connectives and quantifiers, the predicate symbol $<$ with the usual meaning (and maybe some additional numerical predicates), and a special predicate Q_a for each letter a with the meaning: $Q_a x$ is true iff the position x holds the letter a.

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CSClub, St Petersburg, November 20, 2010

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Monadic second order formulas define precisely regular languages (Büchi, 1960), but we would not be able to construct a first order formula defining $(a^2)^*$.

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Given a star-free language L, can one decide the dot-depth of L? In particular, can we decide if the dot-depth of L is 2?

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Dot-depth 1 is known to be decidable (Knast, 1980)

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As discussed in Lecture 4, we may restrict to strongly connected automata.

Here we encounter a small surprise: *every strongly connected aperiodic automaton* $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$ *is synchronizing.*

Indeed, take any $q, q' \in Q$. Since $\mathscr{A}$ is strongly connected, there exists $w \in \Sigma^*$ such that $q \,.\, w = q'$. On the other hand, $\mathscr{A}$ is aperiodic whence there exists a positive integer m such that $q \,.\, w^m = q \,.\, w^{m+1}$. Applying w^m to the equality $q \,.\, w = q'$, we get $q \,.\, w^{m+1} = q' \,.\, w^m$ whence $q \,.\, w^m = q' \,.\, w^m$. Thus, every pair of states can be synchronized, and by Černý's criterion, this ensures that $\mathscr{A}$ is synchronizing.

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Recall that $\frac{n(n-1)}{2}$ is precisely Rystsov's bound for n-state synchronizing automata having a sink. Thus, it remains to prove that every strongly connected aperiodic automaton with n states has a reset word of length $\frac{n(n-1)}{2}$.

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Avraham Trahtman (The Černý conjecture for aperiodic automata, Discrete Math. Theor. Comp. Sci. 9(2) (2007) 3–10) has proved that every synchronizing aperiodic automaton with n states has a reset word of length $\frac{n(n-1)}{2}$ (so less than $(n-1)^2$).

Recall that $\frac{n(n-1)}{2}$ is precisely Rystsov's bound for n-state synchronizing automata having a sink. Thus, it remains to prove that every strongly connected aperiodic automaton with n states has a reset word of length $\frac{n(n-1)}{2}$.

The key observation by Trahtman is that every strongly connected aperiodic automaton admits a non-trivial stable partial order.

Given a DFA $\mathscr{A} = \langle Q, \Sigma, \delta \rangle$, its square $\mathscr{A}^{[2]} = \langle Q \times Q, \Sigma, \delta^{[2]} \rangle$ is defined by $\delta^{[2]}((q,p),a) = \big(\delta(q,a), \delta(p,a)\big)$.

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If $\mathscr{A}$ is synchronizing and strongly connected, then $\mathscr{A}^{[2]}$ has a least strongly connected component $D = \{(q,q) \mid q \in Q\}$. Let K be a strongly connected component immediately following D in the natural order of strongly connected components.

23. Constructing Partial Order

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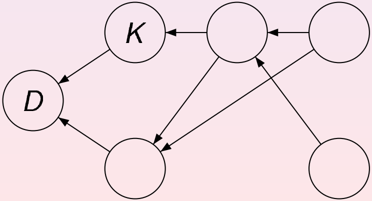

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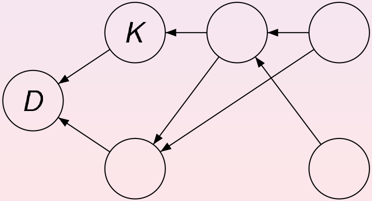

Then $K \cup D$ is a non-trivial stable reflexive relation on Q.

24. $\mathscr{A}$ is aperiodic

Let $\succeq_K$ be the transitive closure of $K \cup D$. It is clear that $\succeq_K$ is non-trivial, stable, reflexive and transitive.

Now we show that $\succeq_K$ is antisymmetric whenever $\mathscr{A}$ is aperiodic.

Suppose that there are $p, q \in Q$ such that $p \neq q$ and $p \succeq_K q \succeq_K p$. Then there is a sequence of $p_0, p_1, \ldots, p_k \in Q$ such that $k > 1$, $p_0 = p = p_k$, $q = p_j$ for some j, $0 < j < k$, and $(p_i, p_{i+1}) \in K$ for all $i = 0, 1, \ldots, k-1$. We choose the shortest such sequence $p_0, p_1, \ldots, p_k$ (over all possible 'obstacles' (p, q) to antisymmetry).

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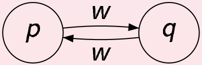

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Thus, $\succeq_K$ is a non-trivial partial order. How does it help?

We denote by π_K the symmetric closure of $\succeq_K$. Clearly, π_K is a congruence of the automaton $\mathscr{A}$.

The quotient automaton $\mathscr{A}/\pi_K$ has $< n$ states and so it has a short reset word v by the induction assumption (we induct on the number of states). In $\mathscr{A}$, the word v sends the whole state set into a single π_K-class, say, T. In order to quickly synchronize T choose a minimum state $q \in T$ and a maximum state $p \in T$ w.r.t. $\succeq_K$ ($p \neq q$ whenever $|T| > 1$). Since $\mathscr{A}$ is strongly connected, there is a word u_1 of length $< n$ such that $p\,.\,u_1 = q$. Then for each $r \in T$ such that $p \succeq_K r$ we have $q = p\,.\,u_1 \succeq_K r\,.\,u_1$ whence $q = r\,.\,u_1$ as q is minimal. If still $|T\,.\,u_1| > 1$, we can take a maximum state $p' \in T\,.\,.\,u_1$ and repeat the process by taking a word u_2 of length $< n$ such that $p'\,.\,u_2 = q$, and so on.

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How long can be a reset word constructed this way?

From the minimum-maximum symmetry it follows that the number of steps is at most $\frac{|T|}{2}$. In the case when $T = Q$ (actually, this is the worst case) we get at most $\frac{n}{2}$ steps and a word of length at most $n - 1$ is added at each step. The resulting reset word is of length at most $\frac{n(n-1)}{2}$. If $|T| = m < n$, then the quotient automaton $\mathscr{A}/\pi_K$ has at most $n - m + 1$ states and we first need a word v of length at most $\frac{(n-m+1)(n-m)}{2}$ to send Q to T and then a word of length at most $\frac{m(n-1)}{2}$ to synchronize T. It remains to calculate that

$$\frac{(n-m+1)(n-m)}{2} + \frac{m(n-1)}{2} \leq \frac{n(n-1)}{2}$$

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28. Weakly Monotonic Automata

In my paper (Synchronizing automata preserving a chain of partial orders, Theor. Comput. Sci. 410 (2009) 3513–3519) Trahtman's theorem has been extended to a larger class automata.

A DFA $\mathscr{A}$ is weakly monotonic of level ℓ if it has a strictly increasing chain of stable binary relations $\rho_0 \subset \rho_1 \subset \cdots \subset \rho_\ell$ satisfying the following conditions:

- ρ_0 is the equality relation;
- for each $i = 1, \ldots, \ell$, the congruence π_{i-1} generated by ρ_{i-1} is contained in ρ_i and the relation ρ_i/π_{i-1} is a partial order on Q/π_{i-1};
- π_ℓ is the universal relation.

This differs from the notion of a generalized monotonic automaton by just dropping the restriction that the order ρ_i/π_{i-1} is linear on each π_i/π_{i-1}-class.

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- every aperiodic automaton is weakly monotonic;
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29. Examples

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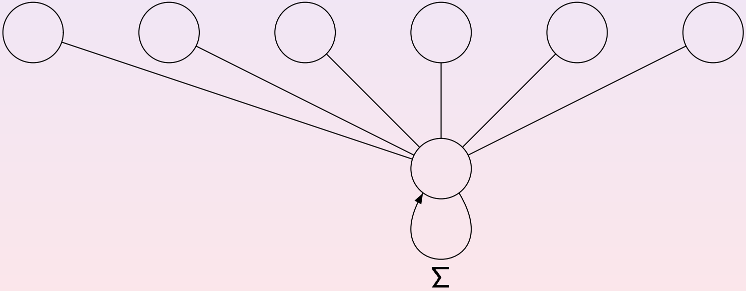

30. WM + SC Automata

- Every weakly monotonic automaton with a strongly connected underlying digraph is synchronizing. (A non-trivial generalization of the corresponding result for aperiodic automata.)

- Every weakly monotonic automaton with a strongly connected underlying digraph and n states has a reset word of length $\leq \left\lfloor \frac{n(n+1)}{6} \right\rfloor$. (This upper bound is new even for the aperiodic case – recall that Trahtman's bound was 3 times higher, namely, $\frac{n(n-1)}{2}$.)

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A bad news is that there are no matching lower bounds for the upper bounds just discussed. No aperiodic (even no weakly monotonic) strongly connected automaton is known for which the minimum length of reset words would be greater than or equal to the number of states.

Let $C_{SCA}(n)$ denote the restriction of the Černý function to the class of all strongly connected aperiodic automata, that is, $C_{SCA}(n)$ is the maximum length of shortest reset words for strongly connected aperiodic automata with n states. Then our current knowledge can be summarized as follows:

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$$n - 1 \leq C_{SCA}(n) \leq \left\lfloor \frac{n(n+1)}{6} \right\rfloor \quad \text{(Volkov, 2009).}$$

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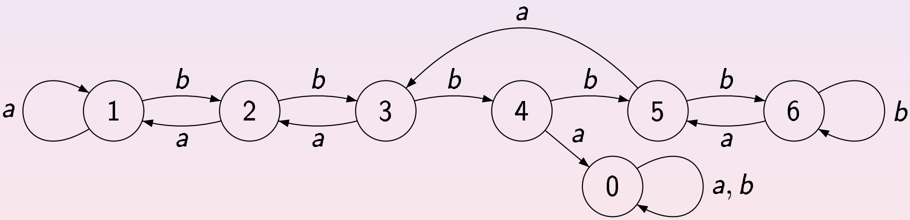

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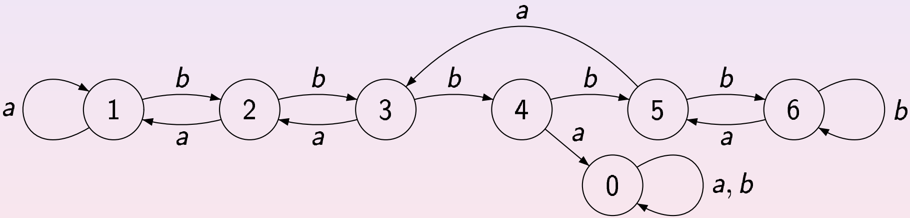

This is the first automaton in Ananichev's series that yields the best known lower bound for $C_A(n)$.

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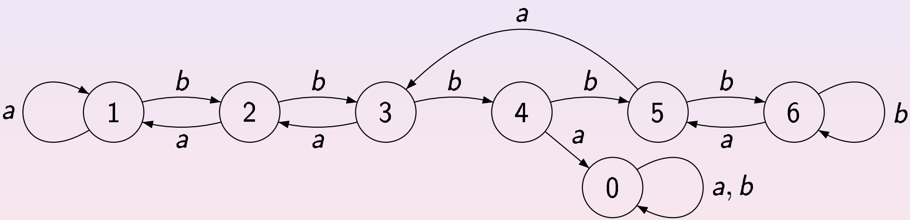
$$\text{(Ananichev, 2010)}\quad n+\left\lfloor\frac{n}{2}\right\rfloor-2 \le C_A(n) \le \frac{n(n-1)}{2}\quad \text{(Trahtman, 2007).}$$



This is the first automaton in Ananichev's series that yields the best known lower bound for $C_A(n)$. It has 7 states and its shortest reset word is a^4b^3a of length $7+\left\lfloor\frac{7}{2}\right\rfloor-2=8$.