Similarity Search

Largely based on a paper joint with

Alexandr Andoni (Columbia), Piotr Indyk (MIT),

Thijs Laarhoven (TU Eindhoven) and Ludwig Schmidt (MIT)

Near Neighbor Search

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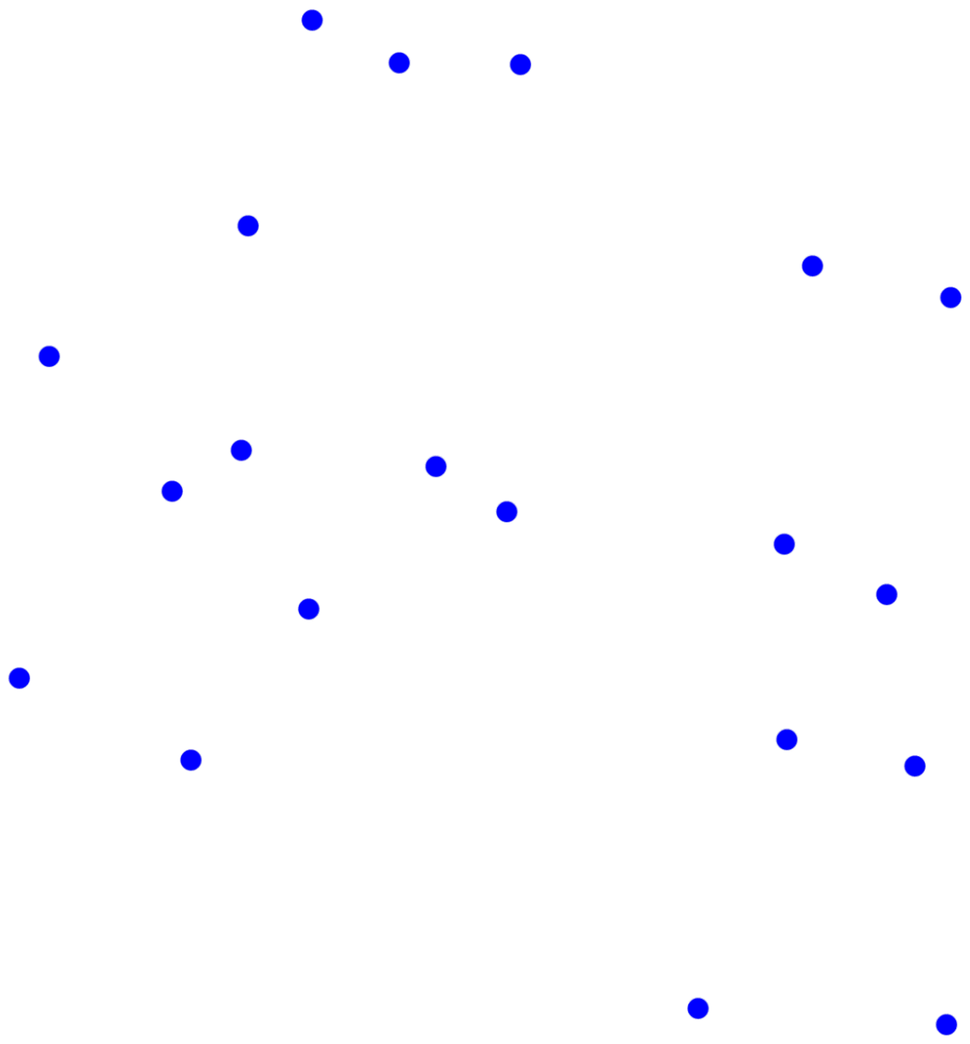
- **Dataset:** **n** points in $\mathbf{R}^d$, $\mathbf{r > 0}$



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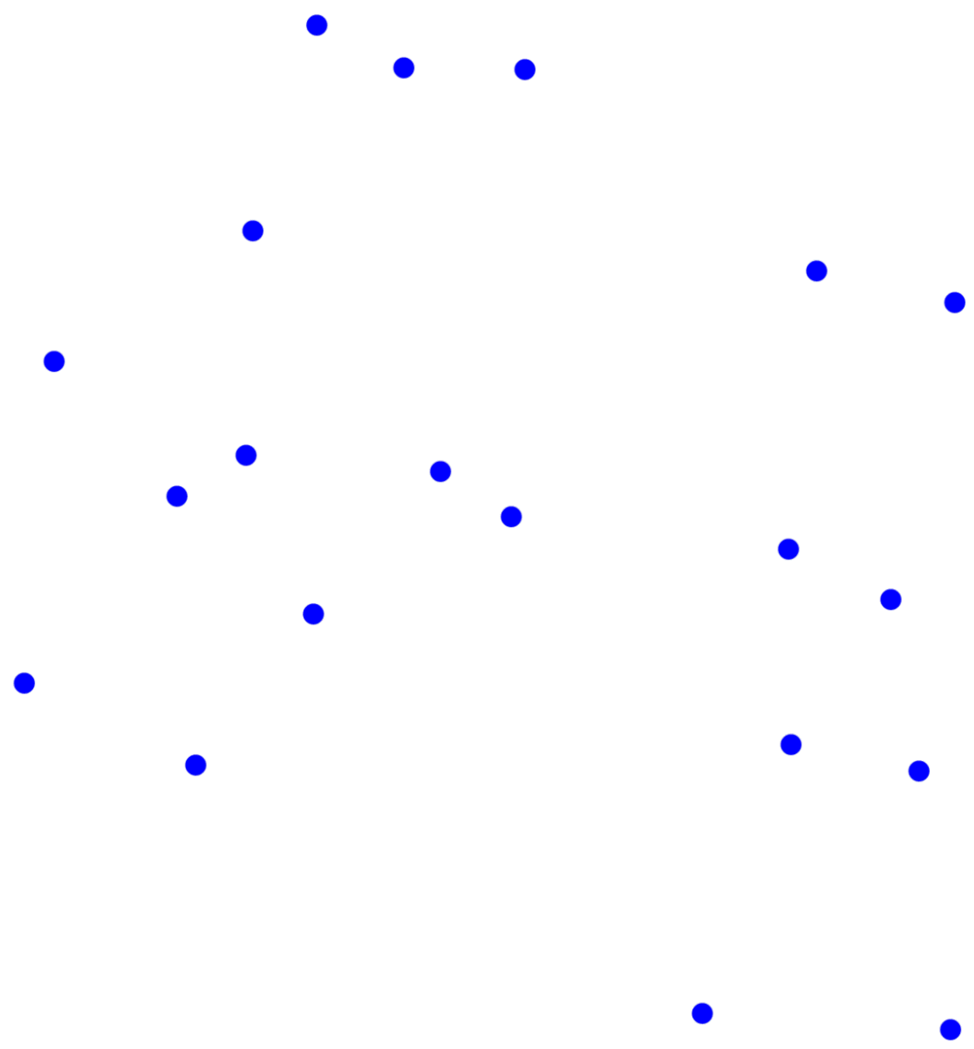

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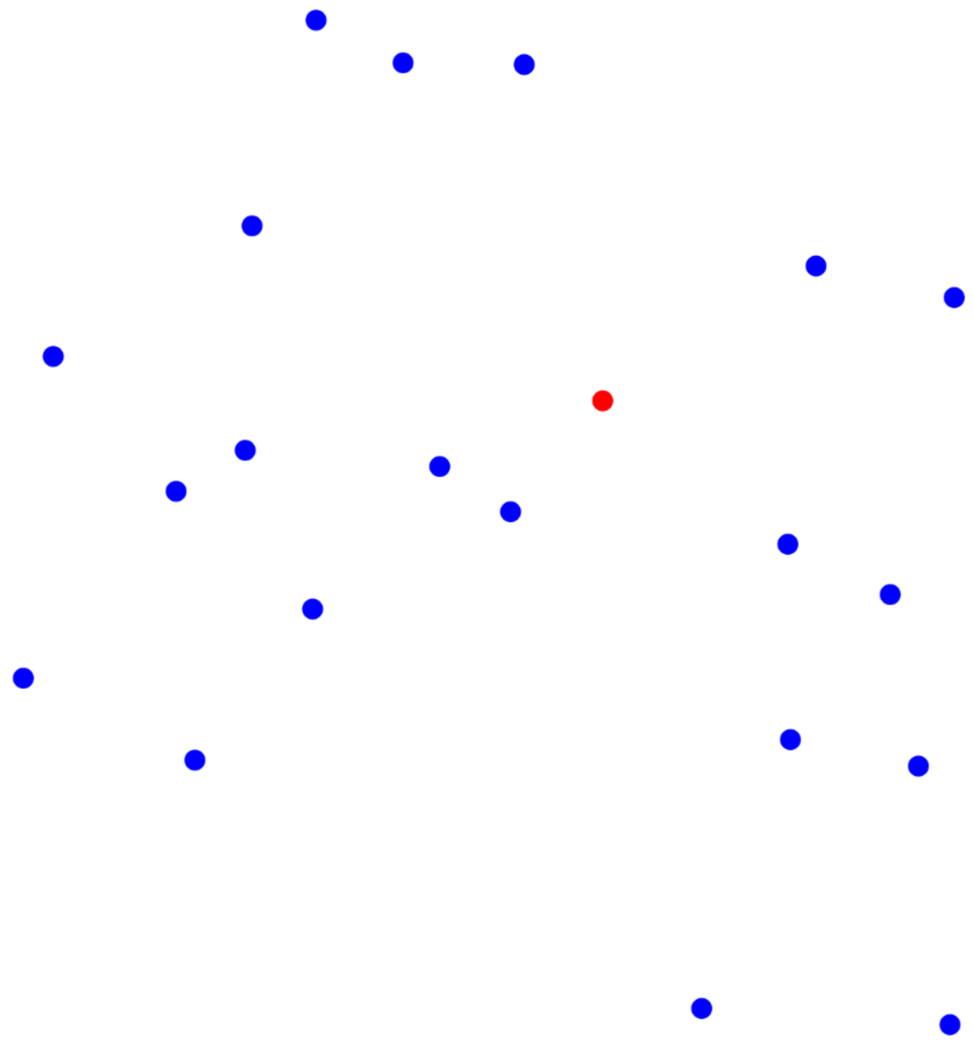

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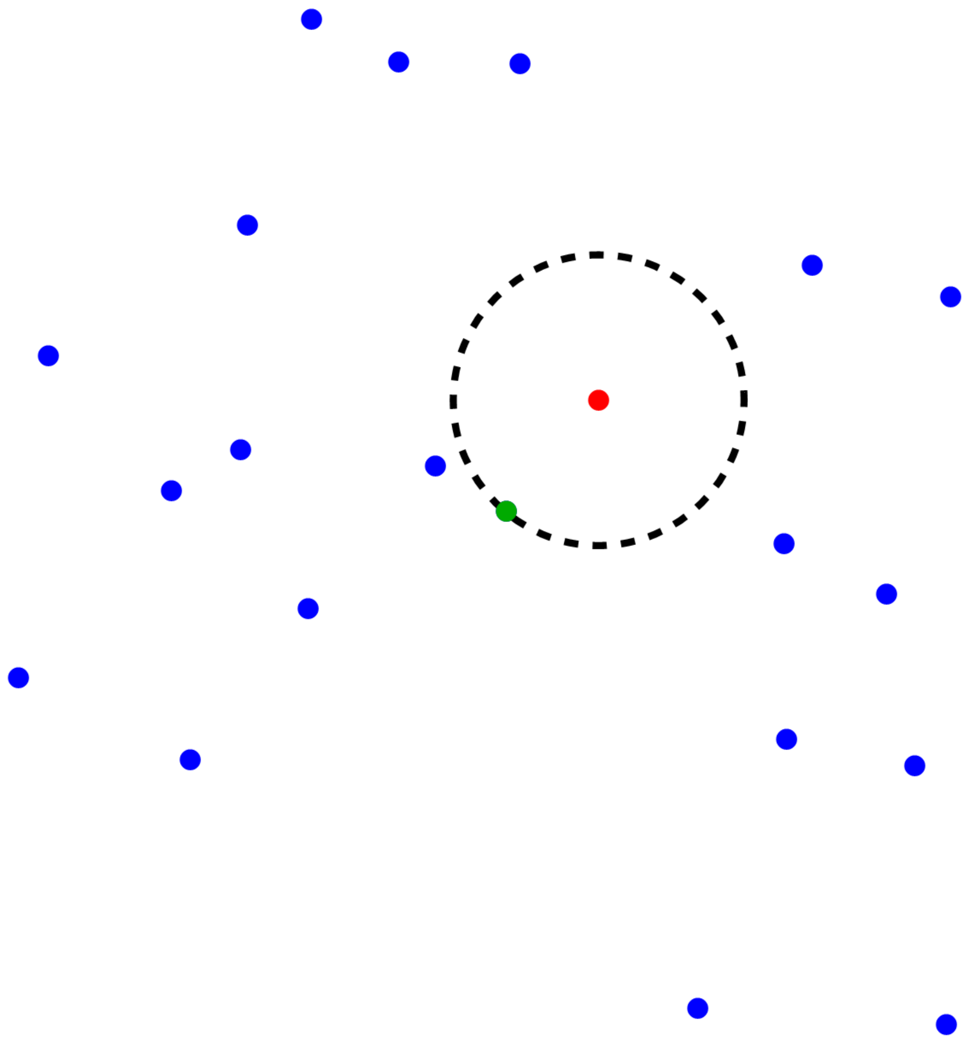

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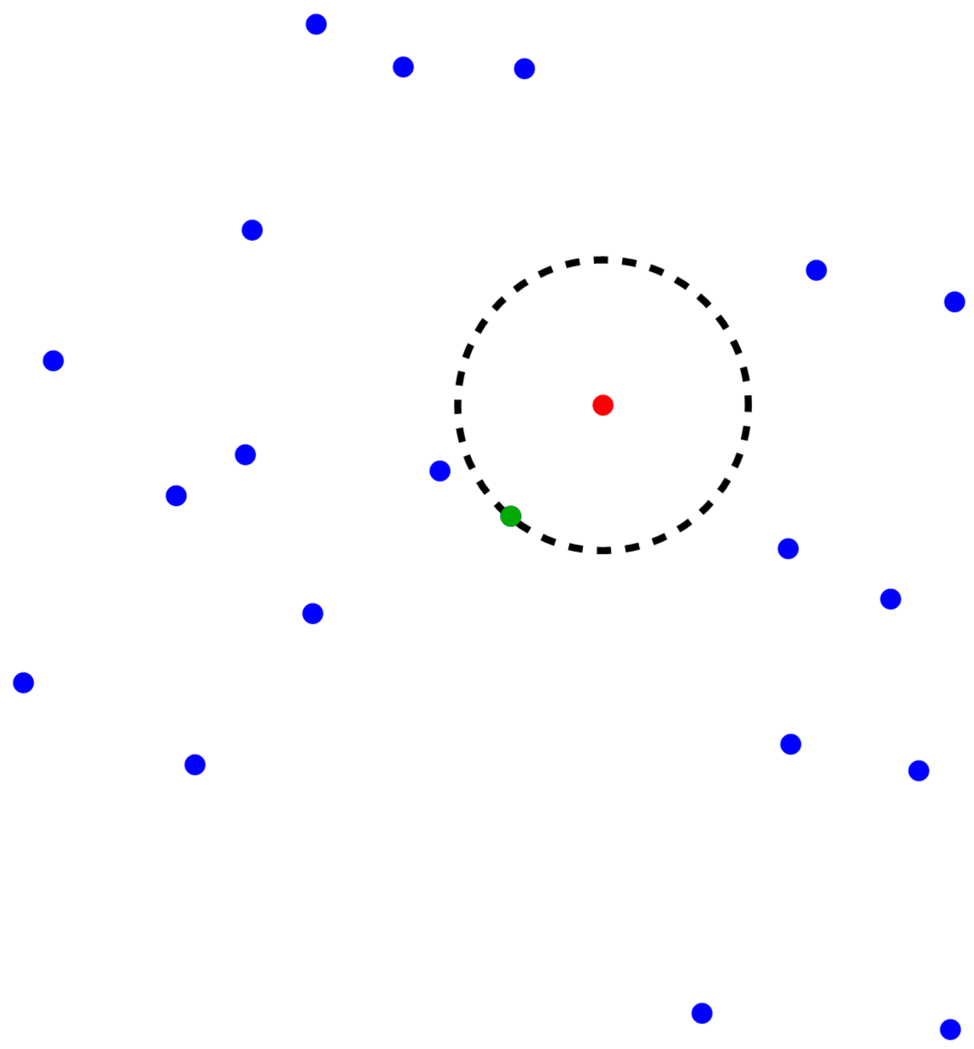

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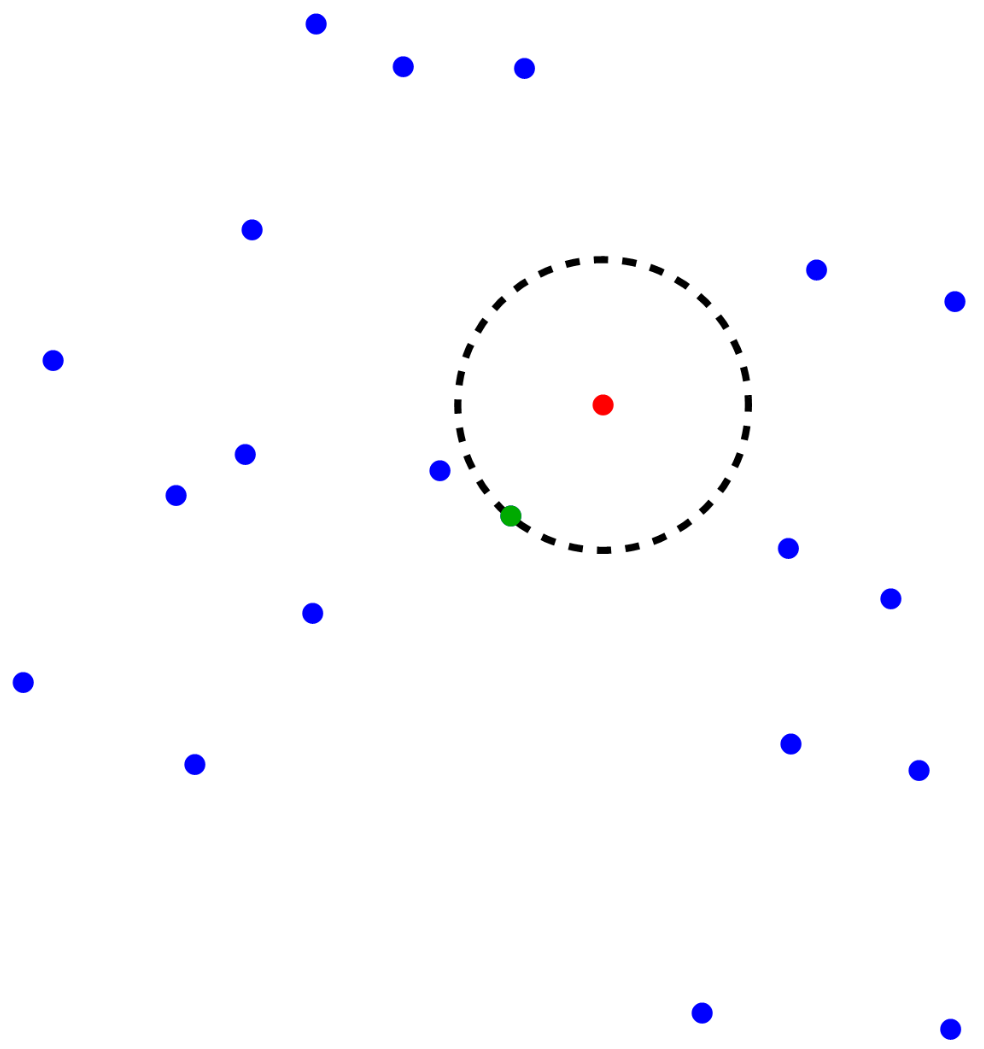

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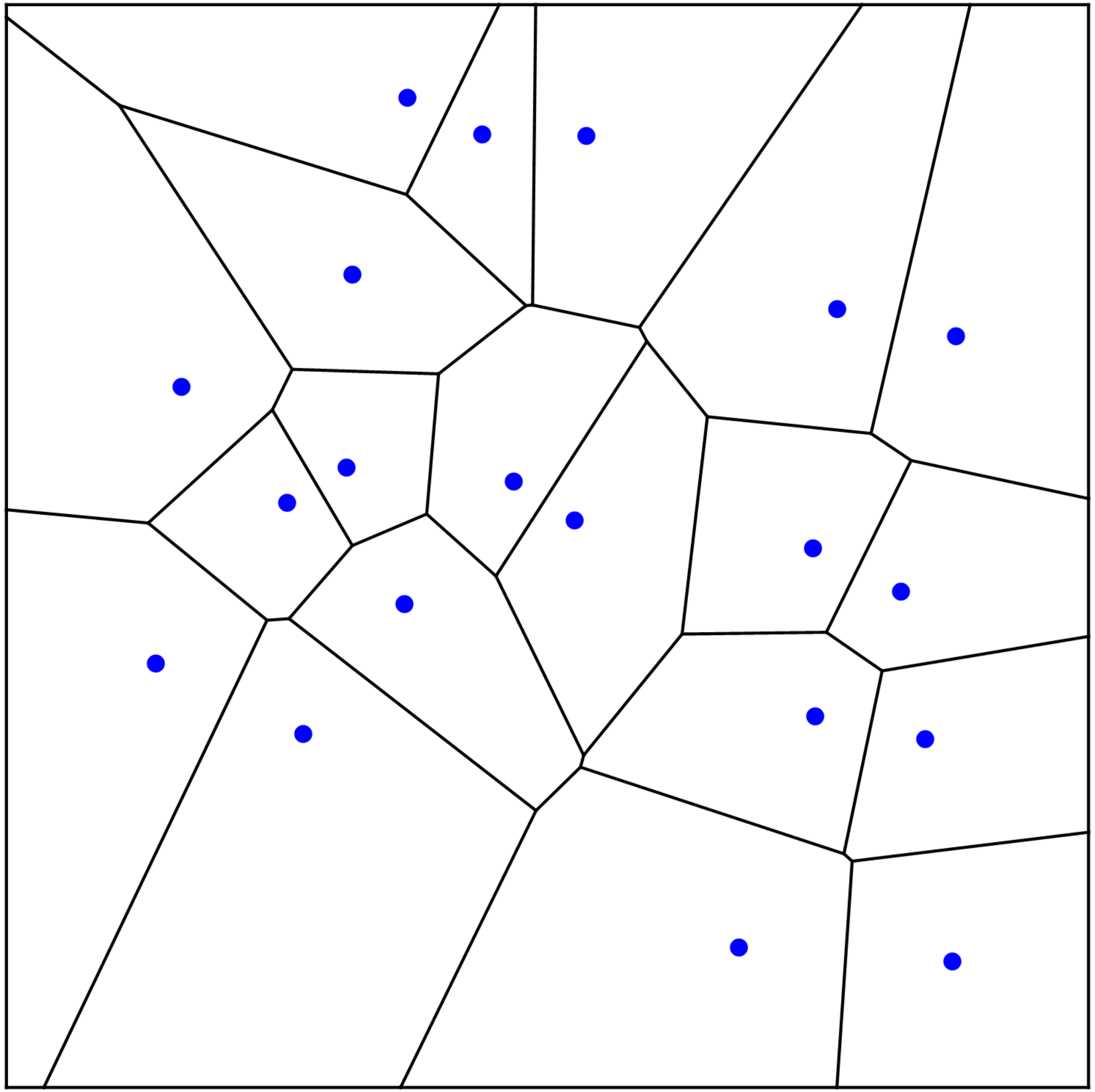

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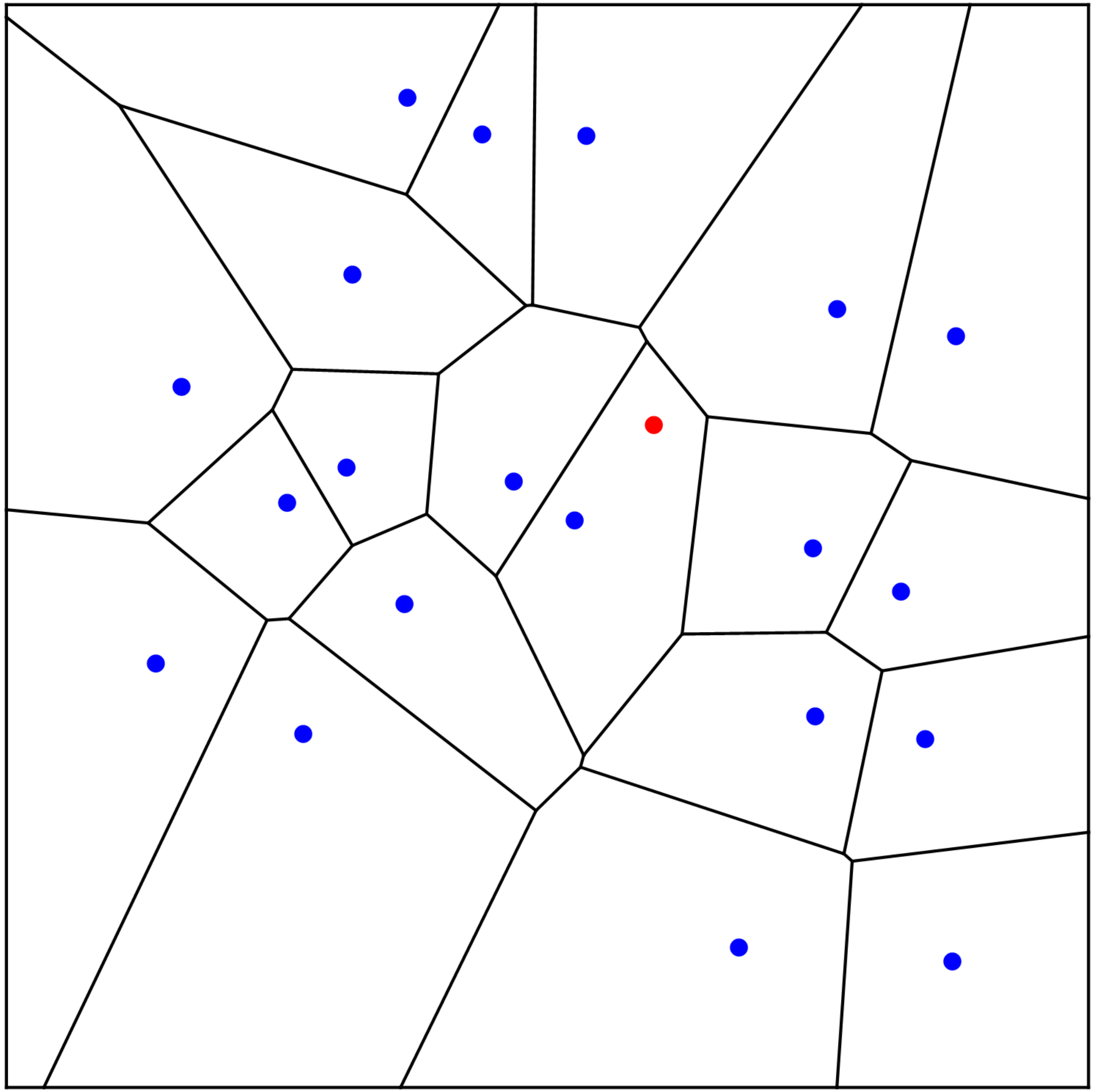

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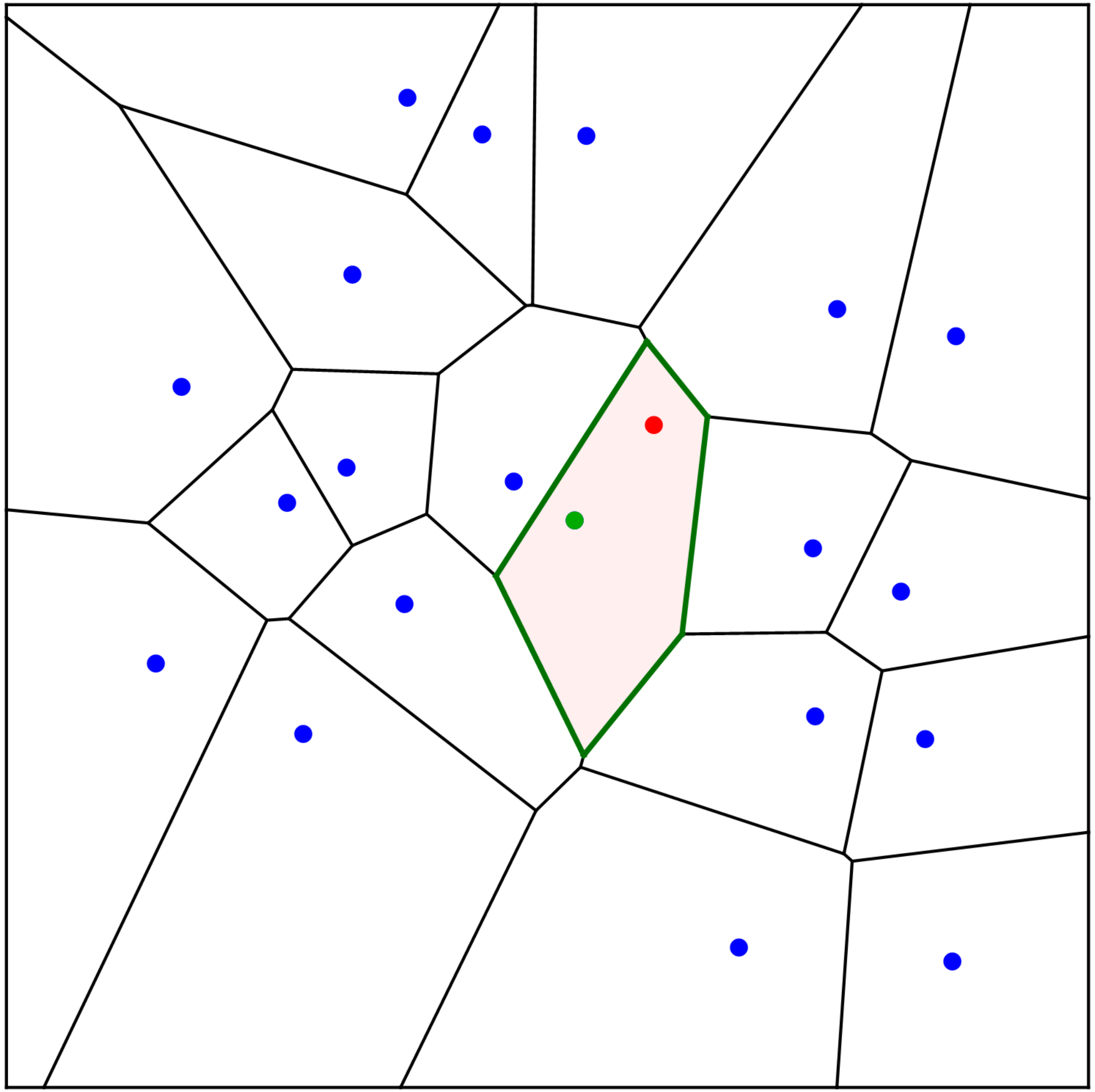

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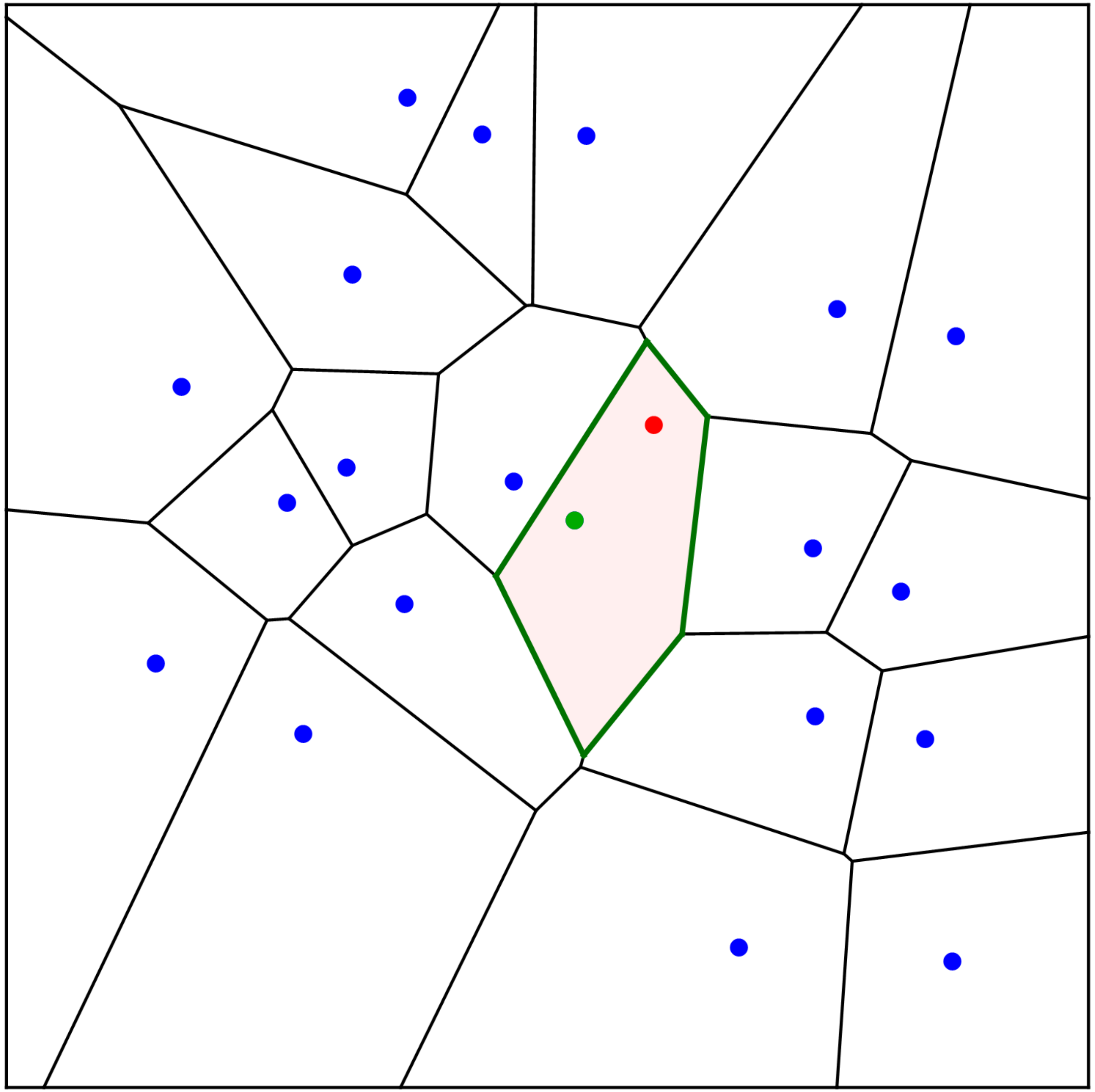

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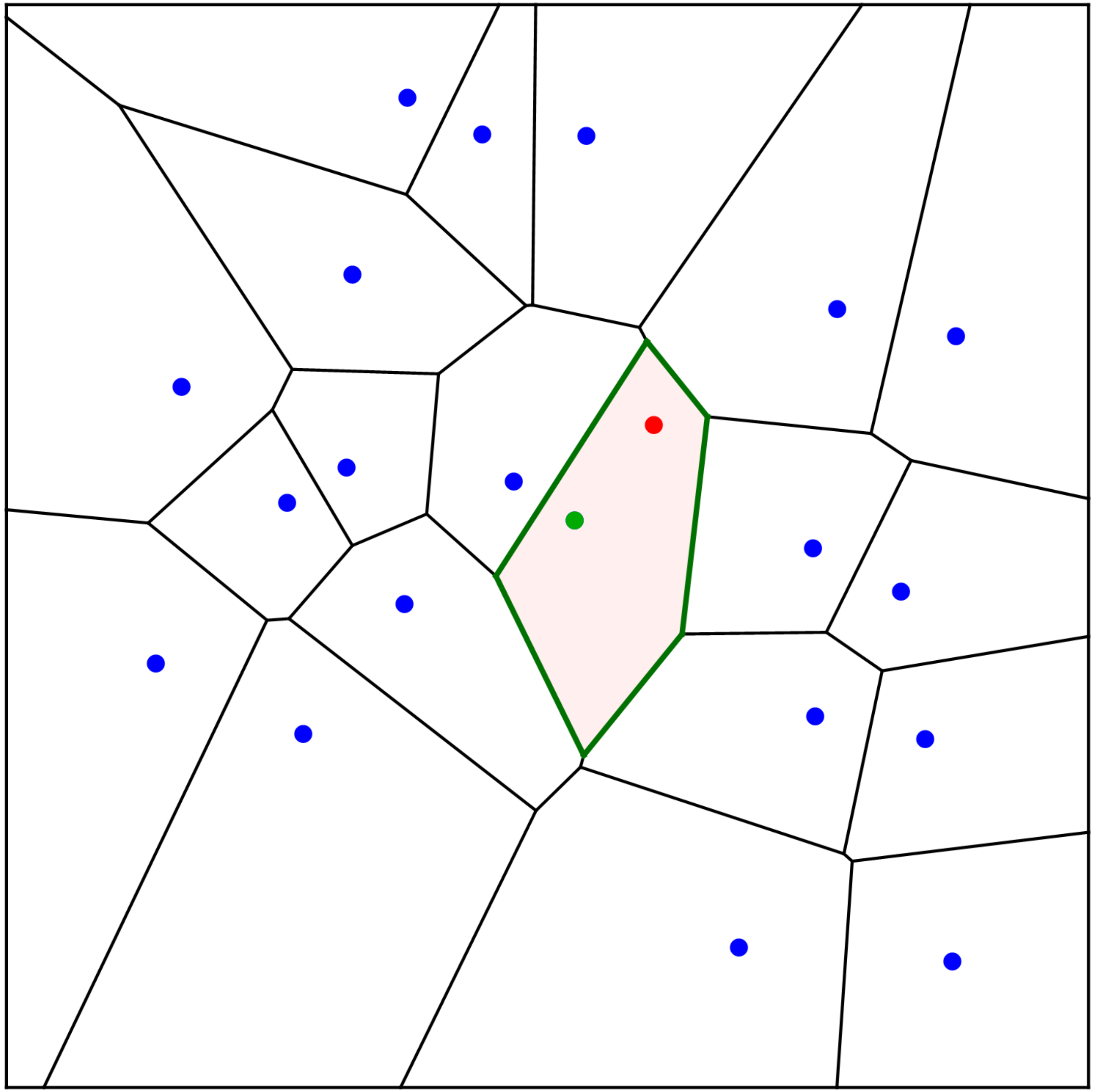
- Most of the applications are in high dimensions



Detour: closest pair in *low* dimensions

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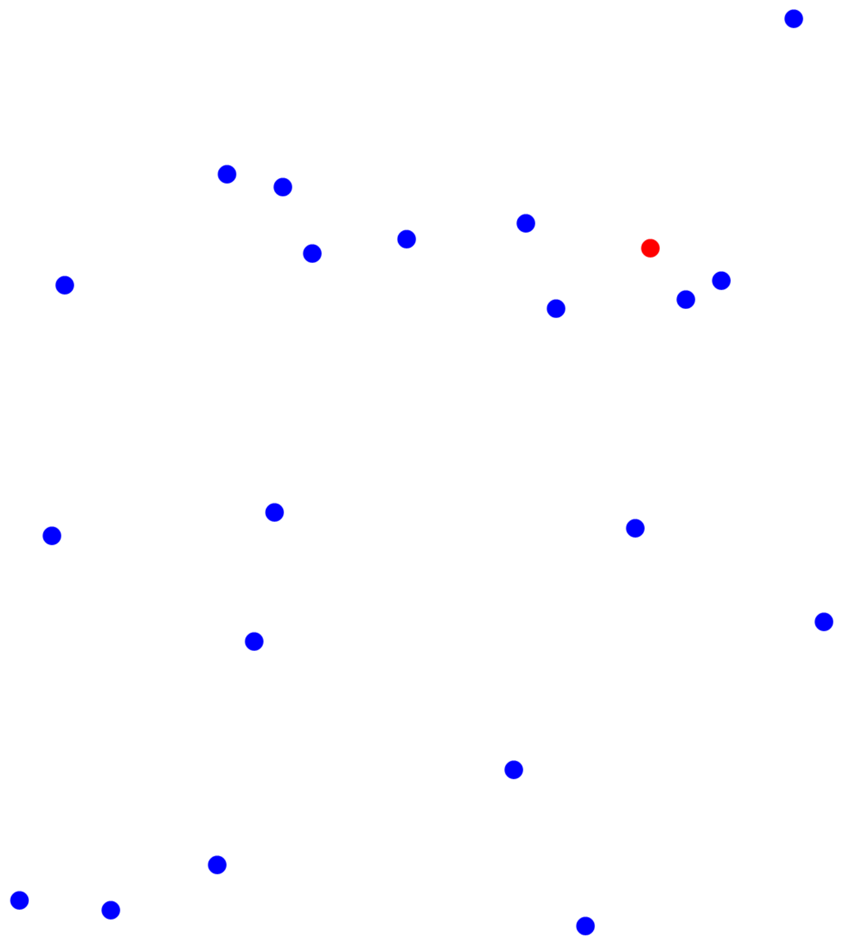

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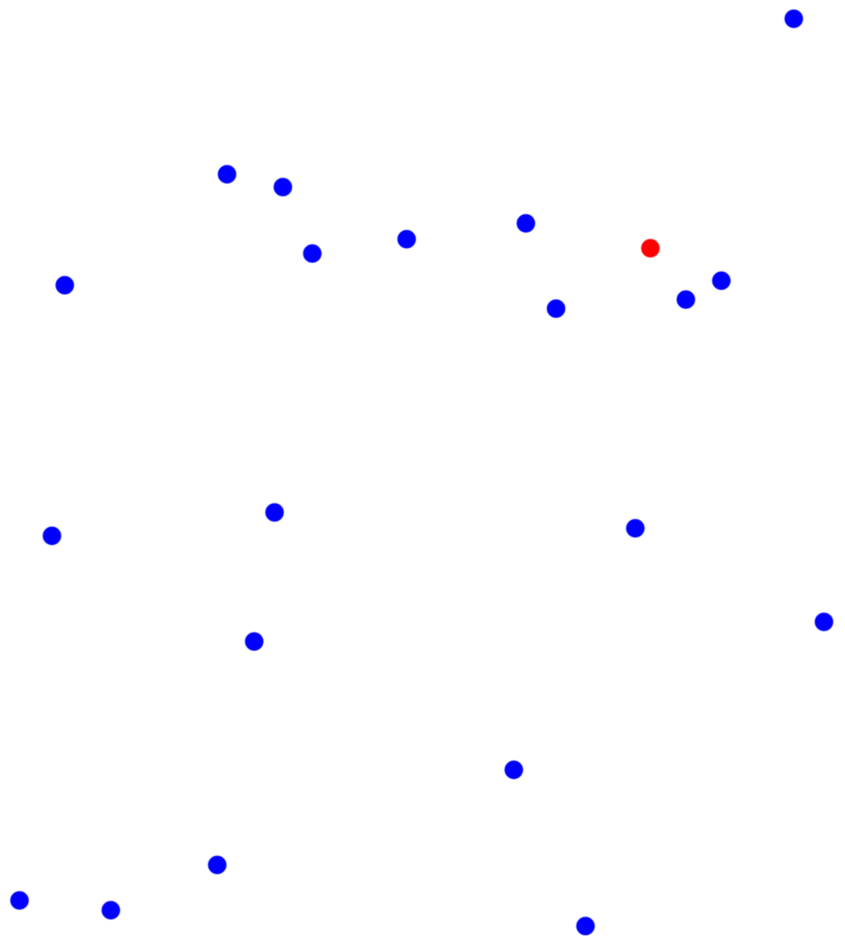

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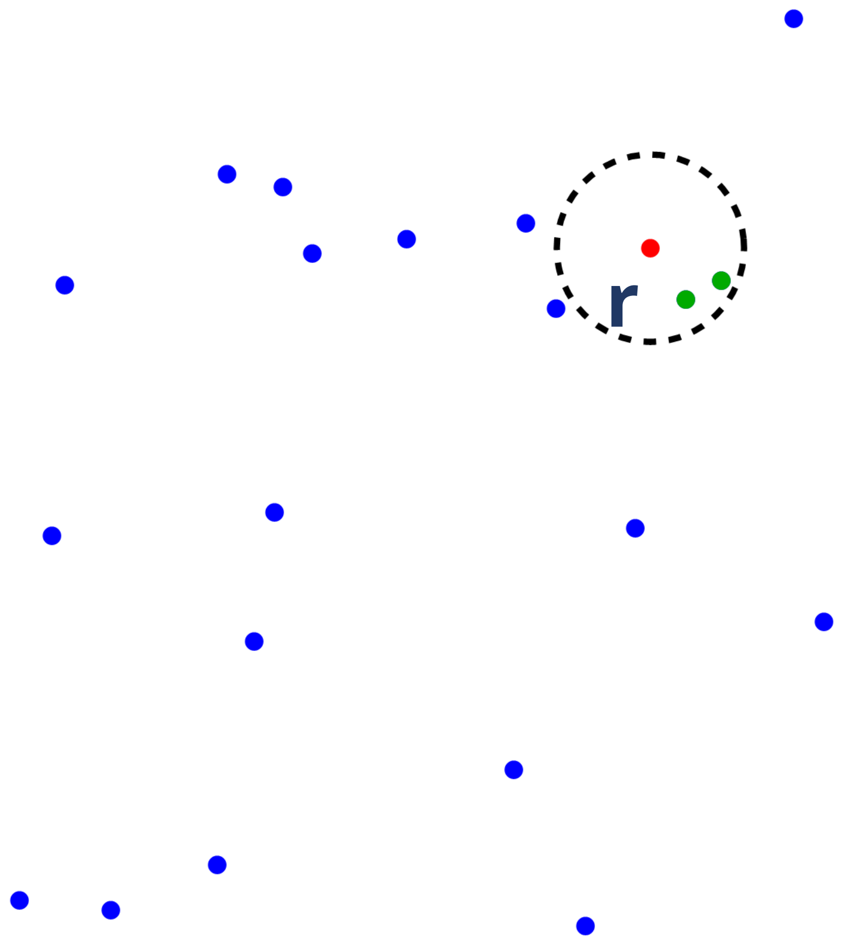

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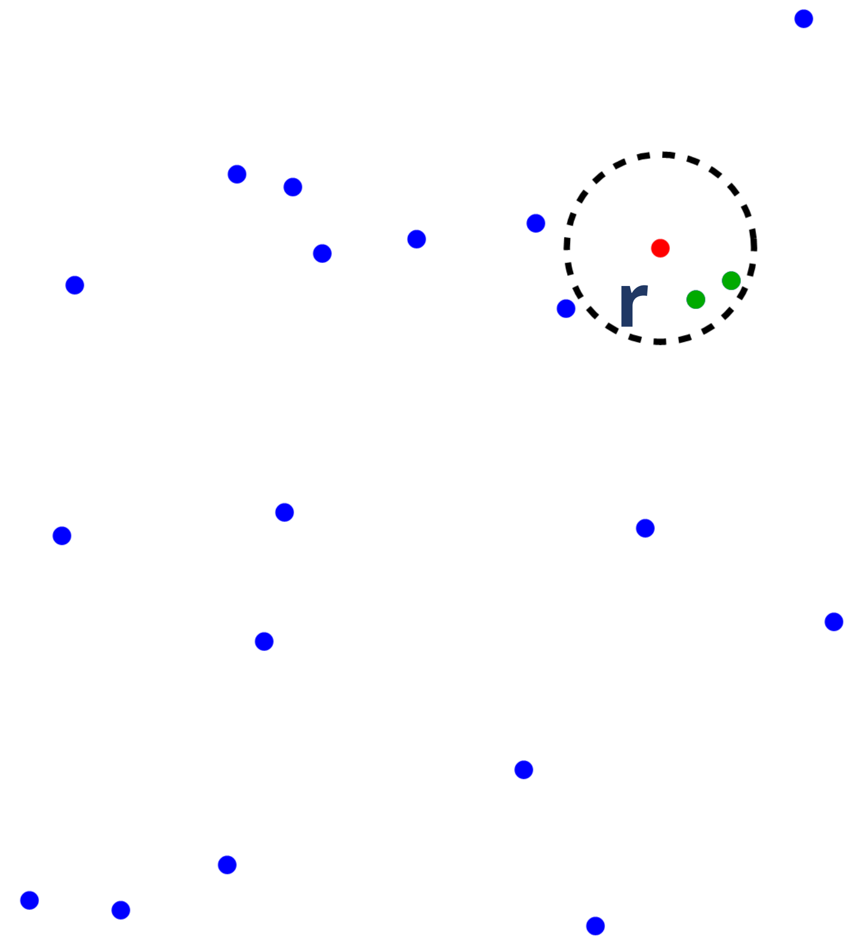

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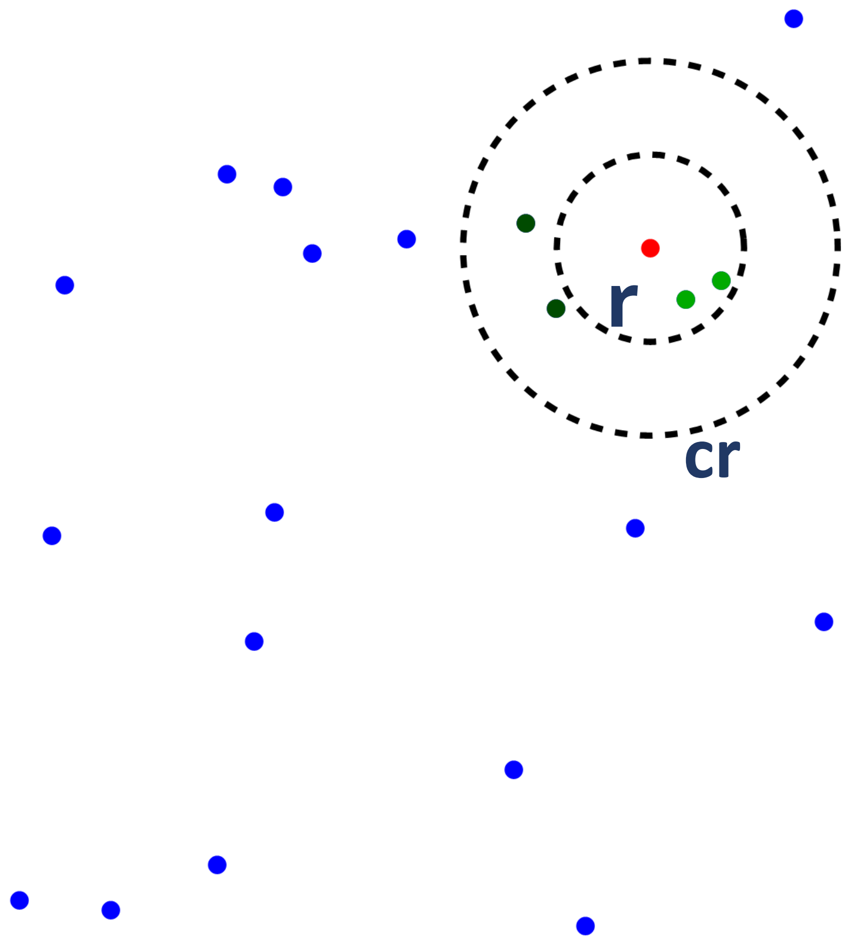

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- Optimization: Coordinate Descent **[Dhillon, Ravikumar, Tewari 2011]**, Stochastic Gradient Descent **[Hofmann, Lucchi, McWilliams 2015]** etc

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- Why interesting?
- **In theory:** can reduce general case to the spherical case **[Andoni, R 2015]**
- **In practice:**
  - Cosine similarity is widely used
  - Oftentimes, can *pretend* that the dataset lies on a sphere

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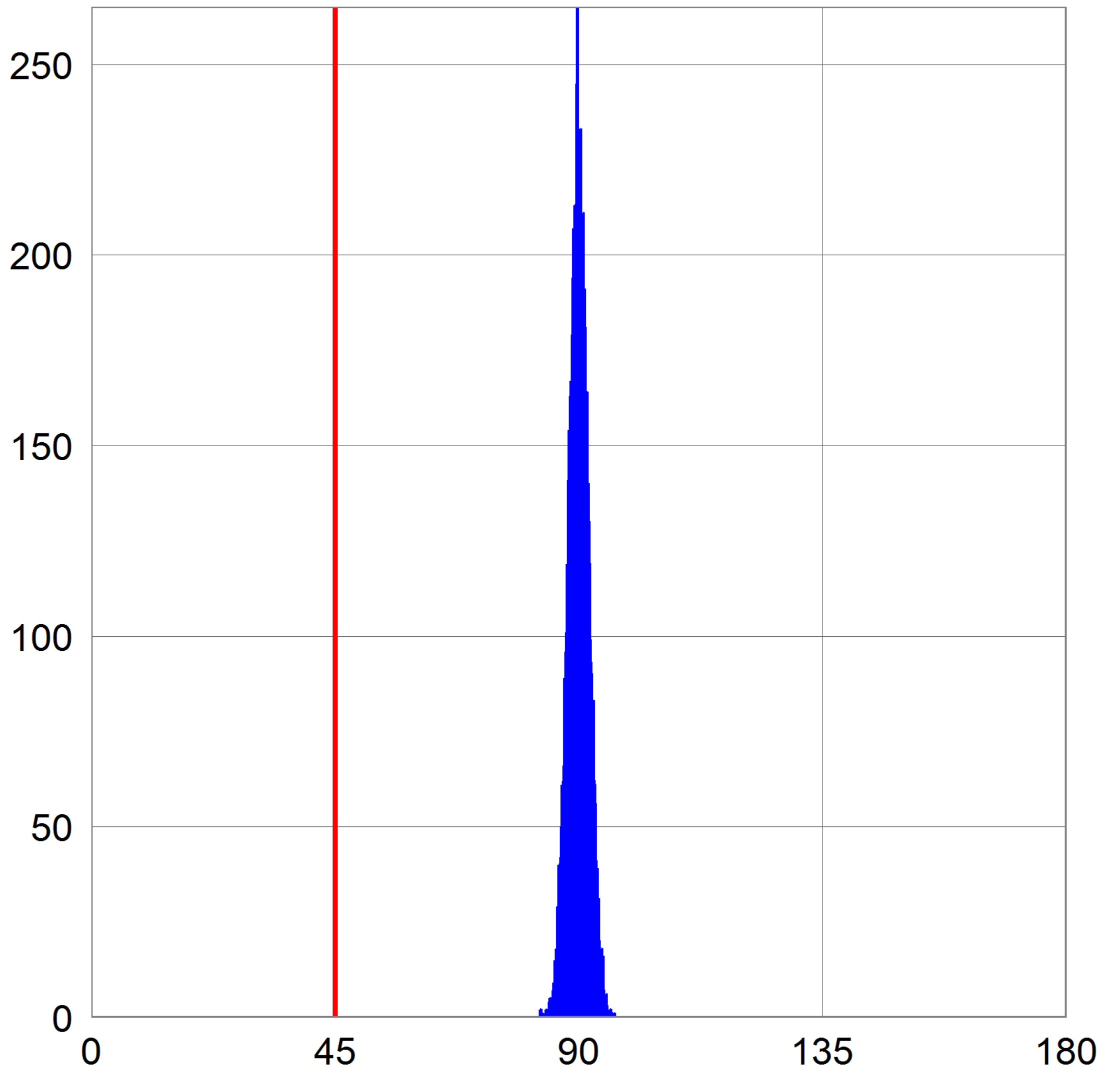

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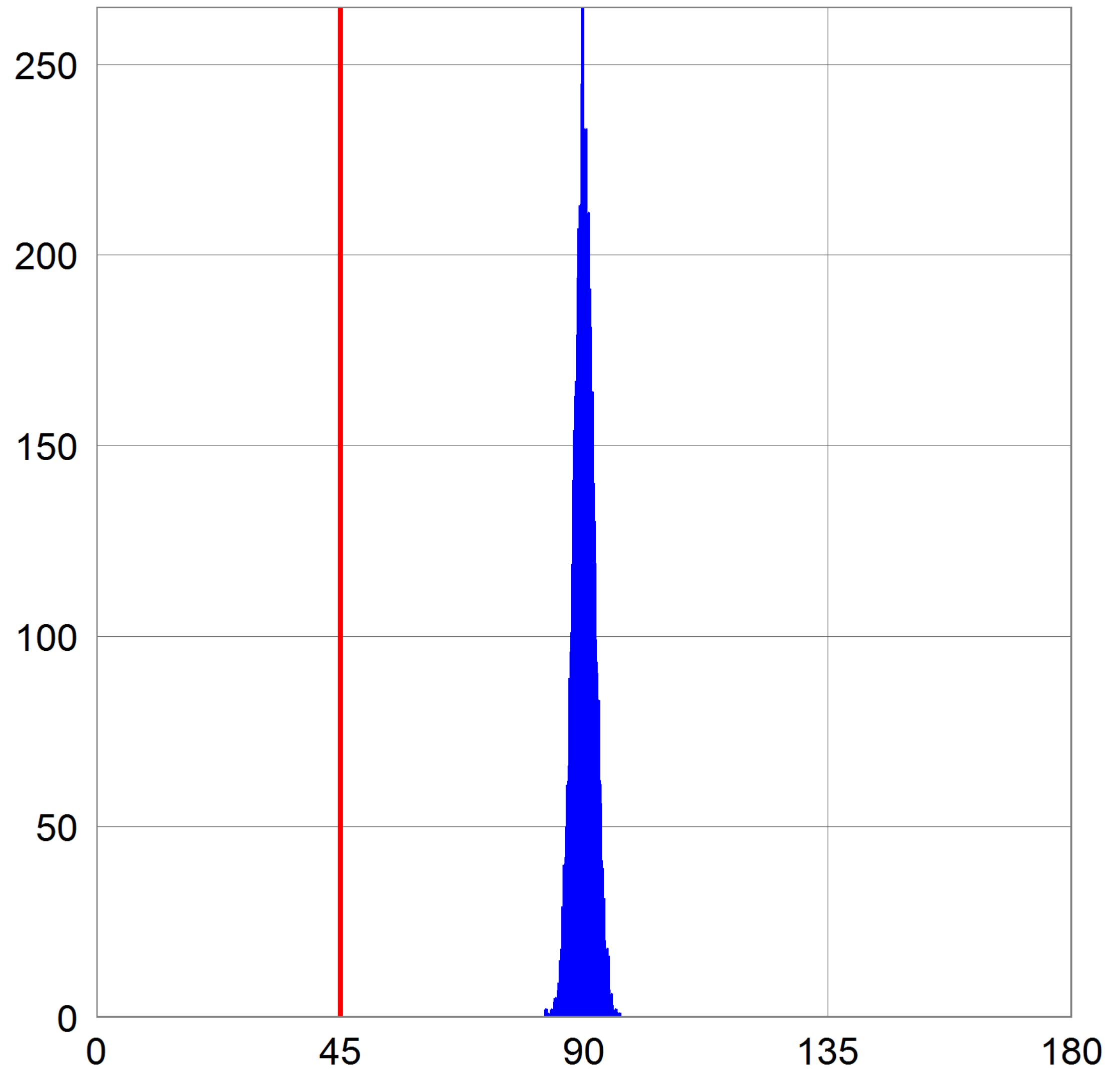
- Instructive case to think about
  - **[Andoni, R 2015]:** a (delicate) reduction from general to random
  - Concentration of angles around **90** degrees happens in practice



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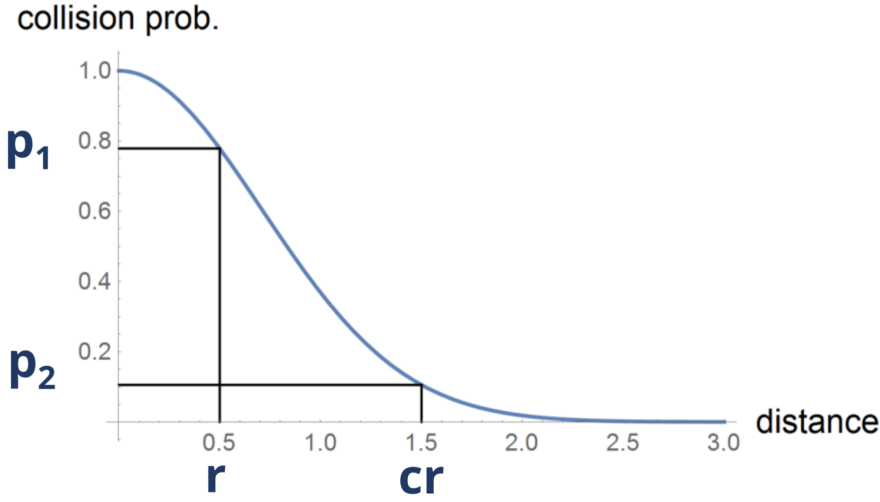

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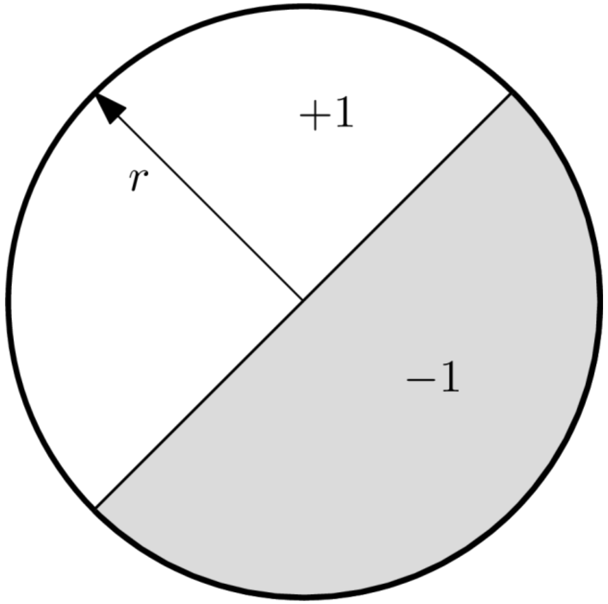

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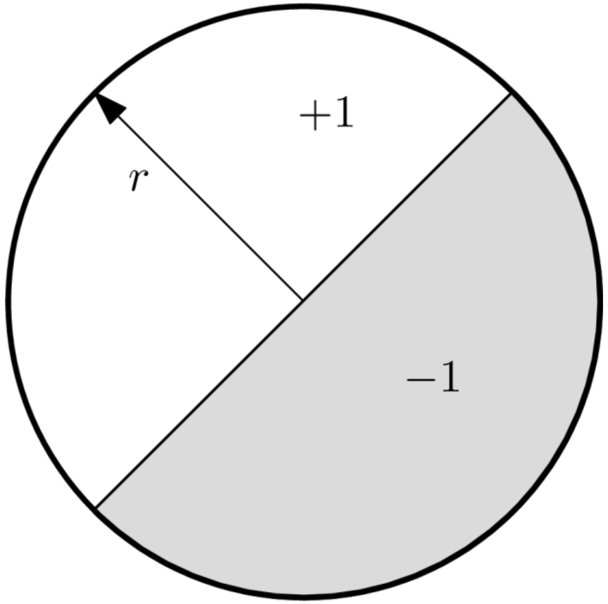

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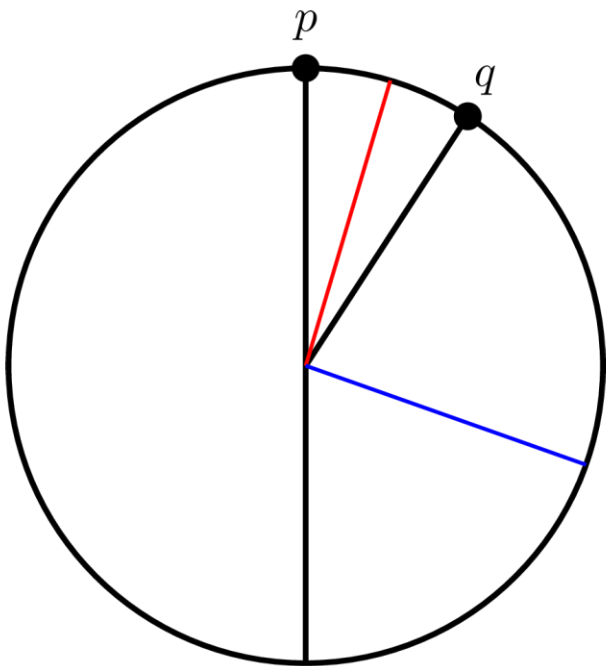

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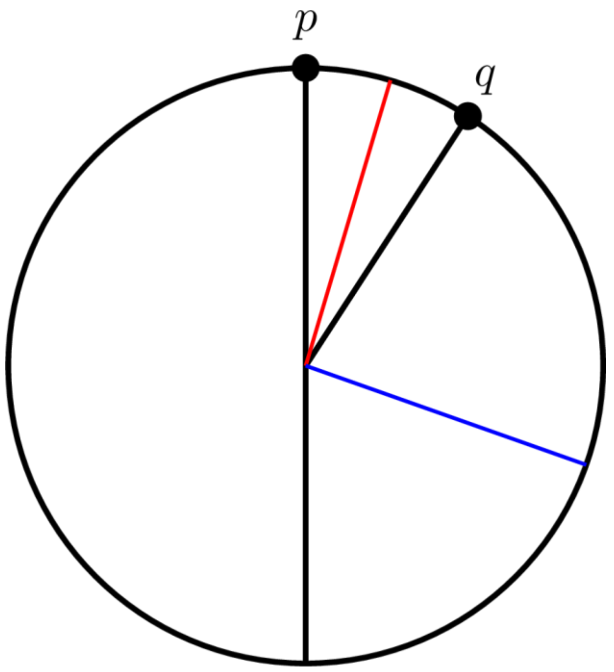

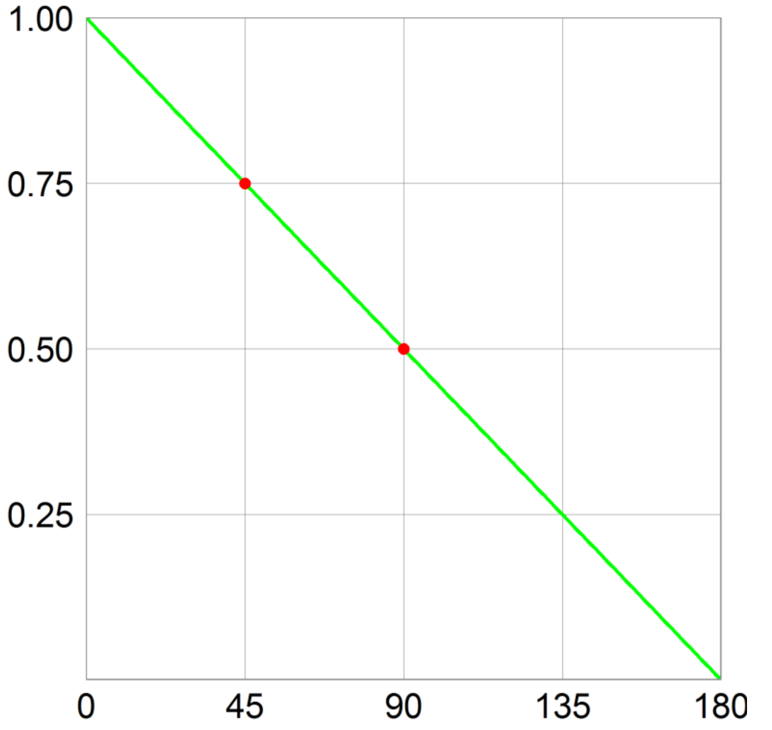

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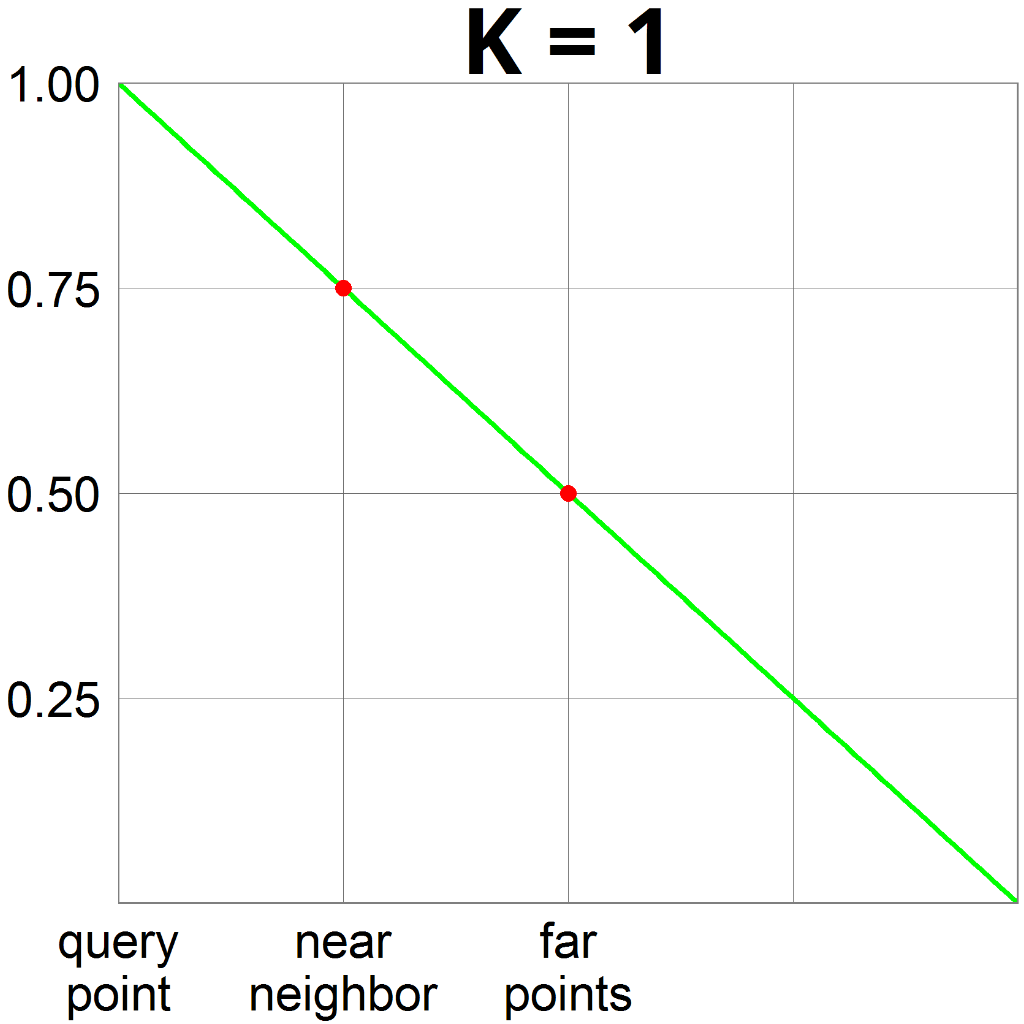

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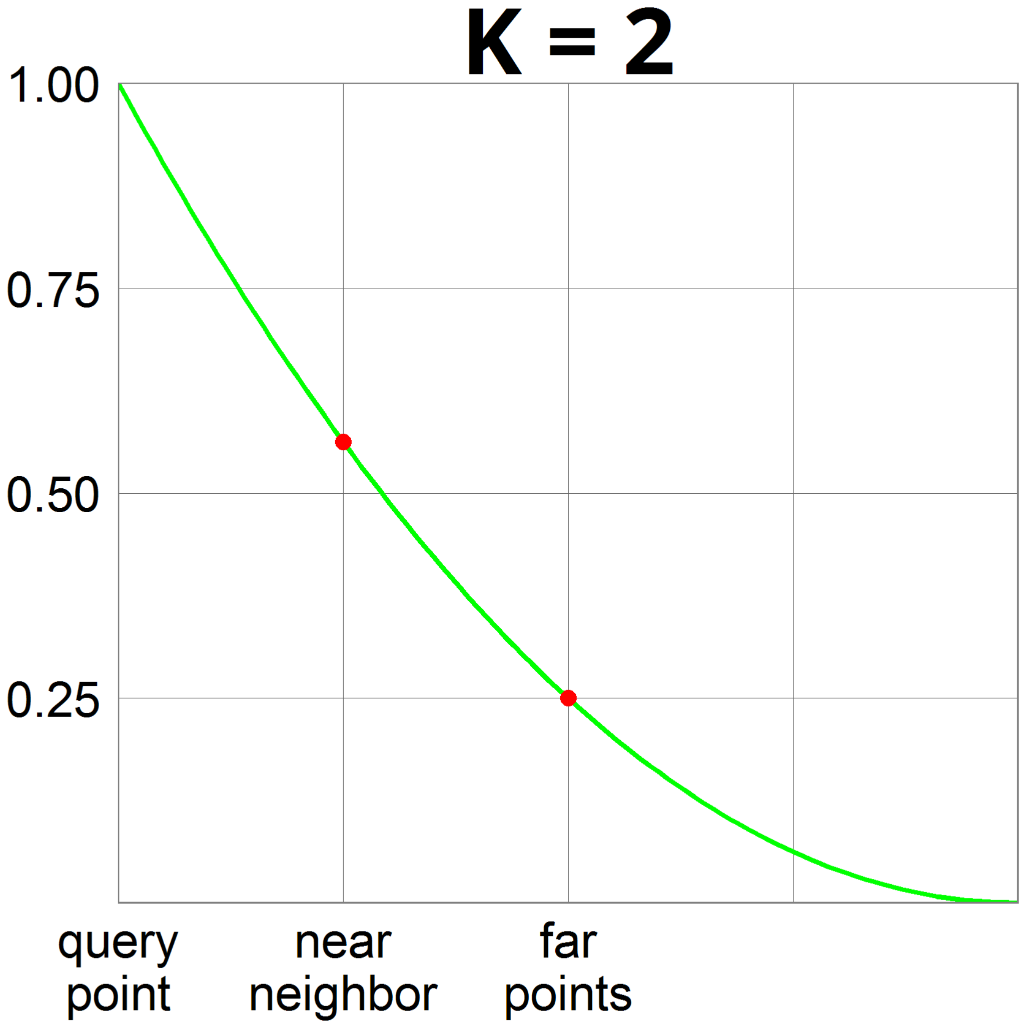

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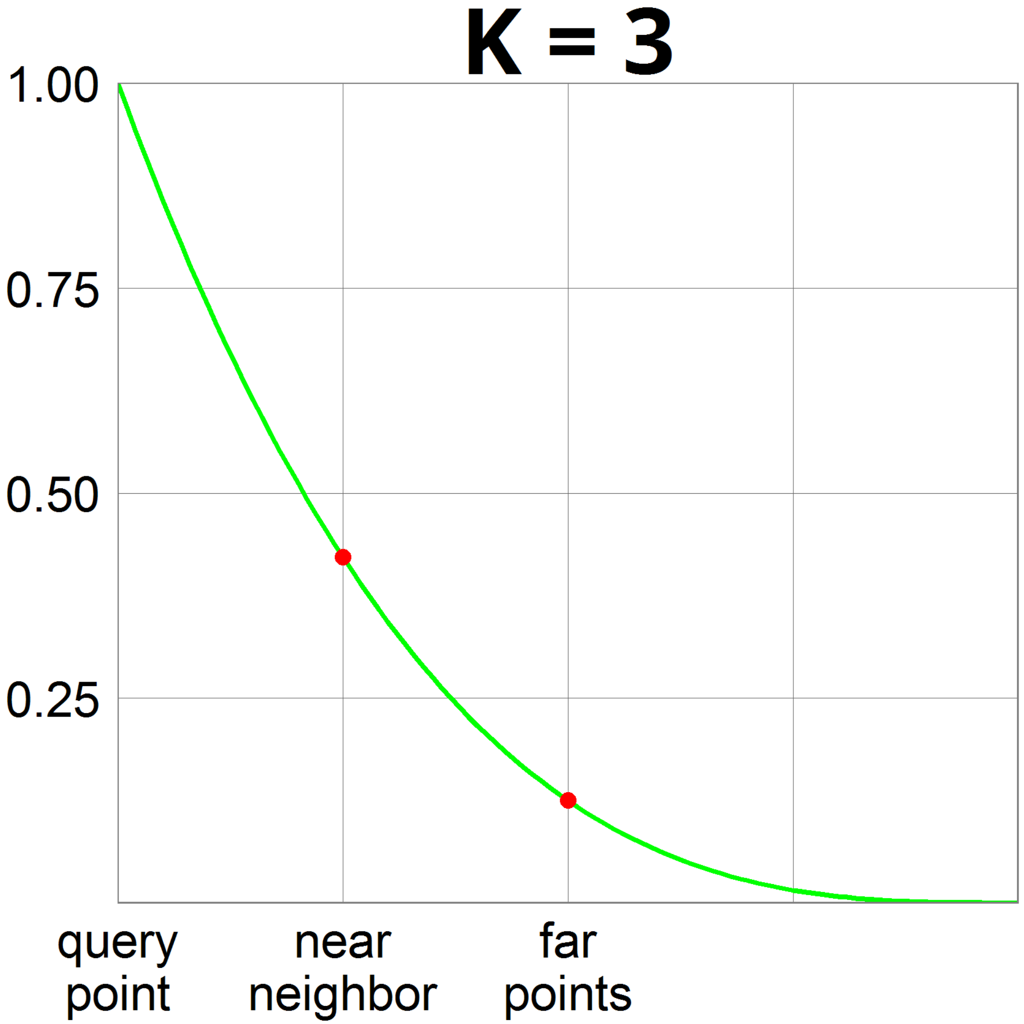

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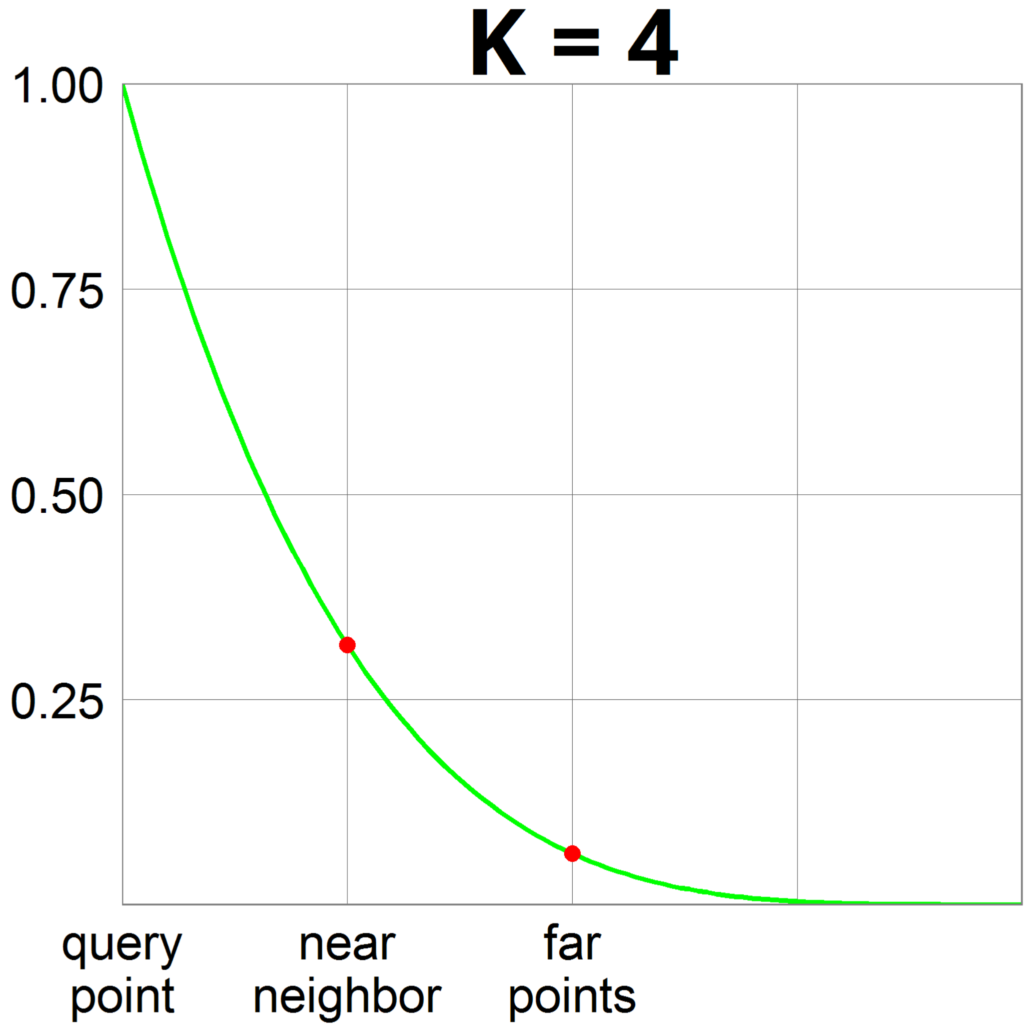

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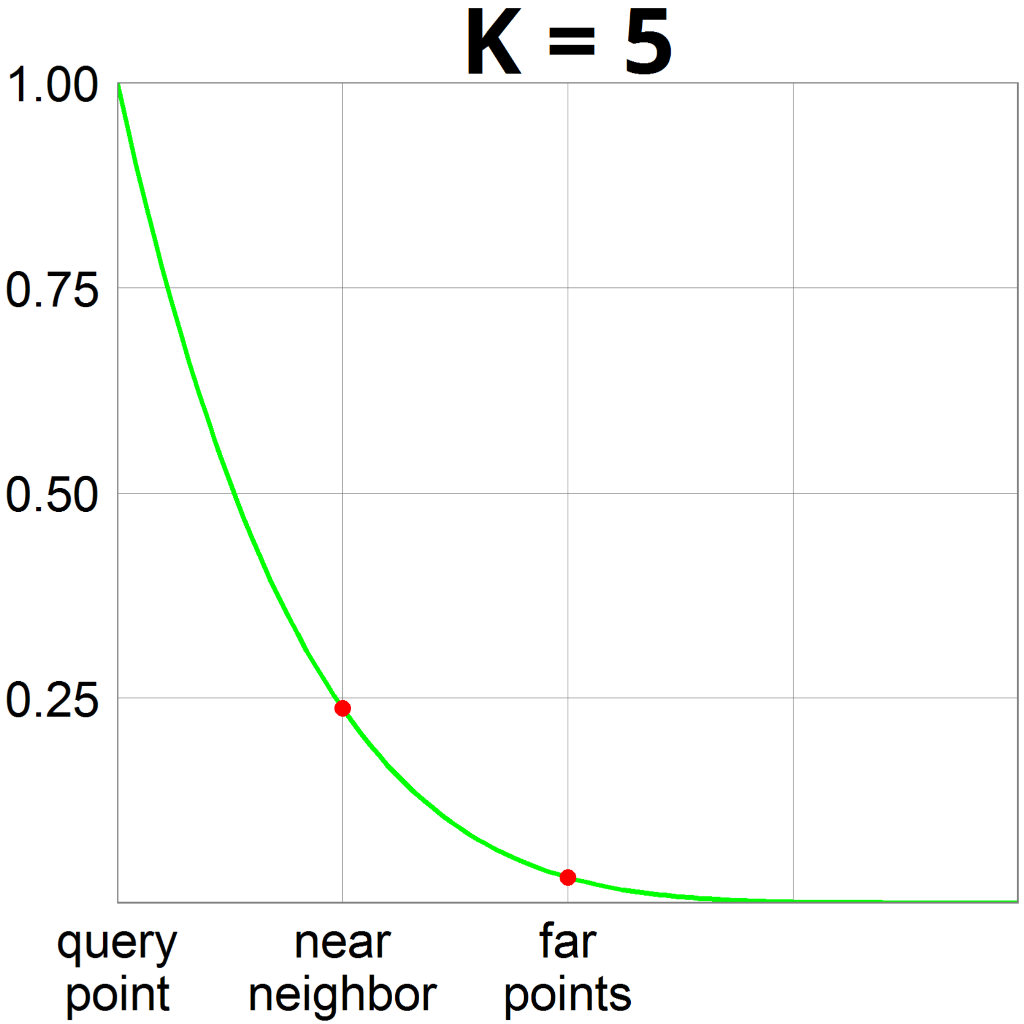

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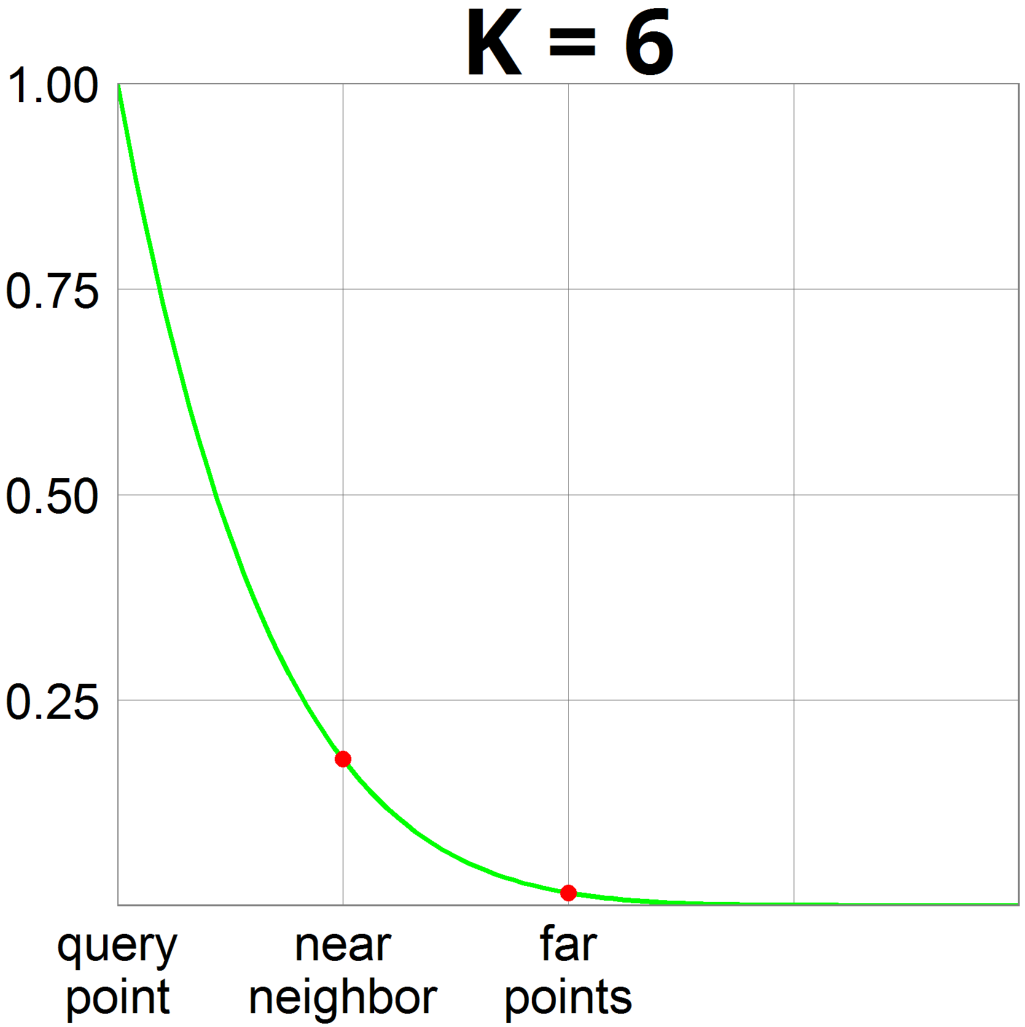

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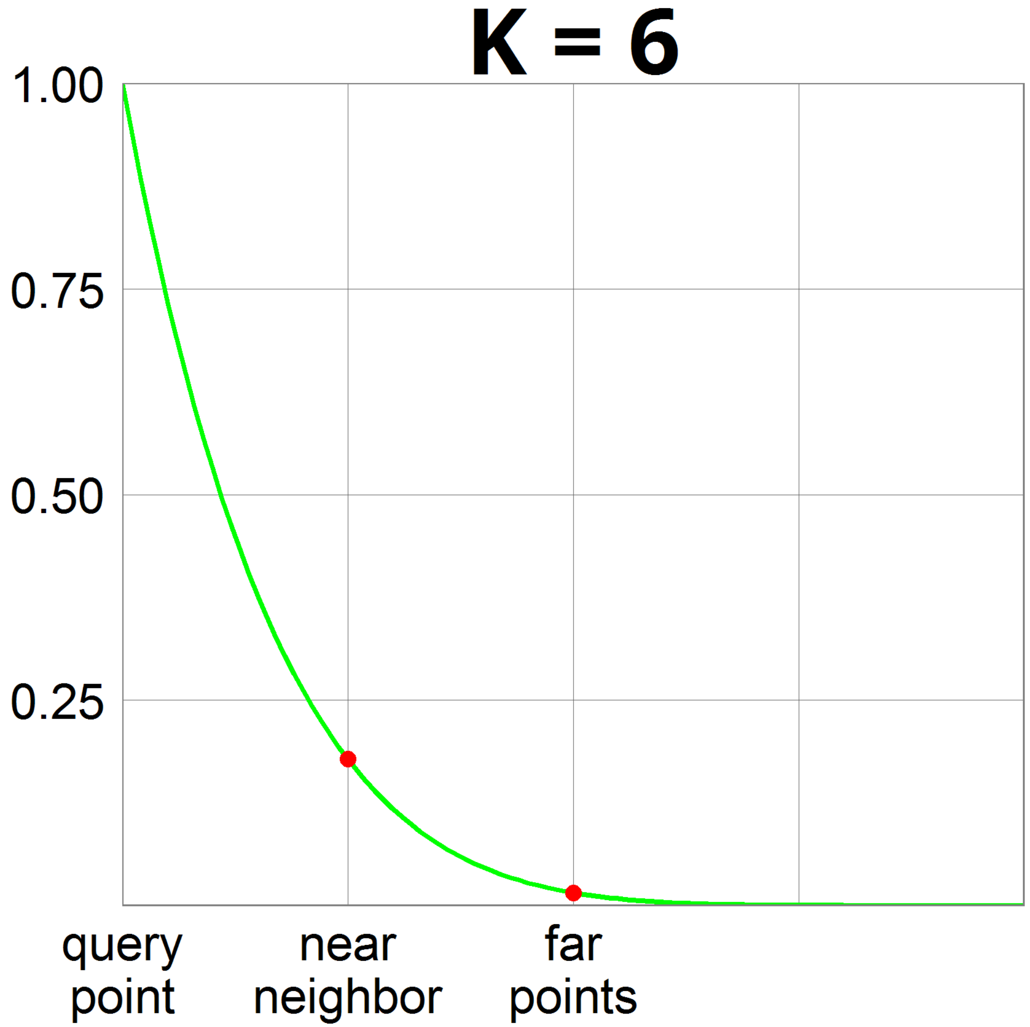

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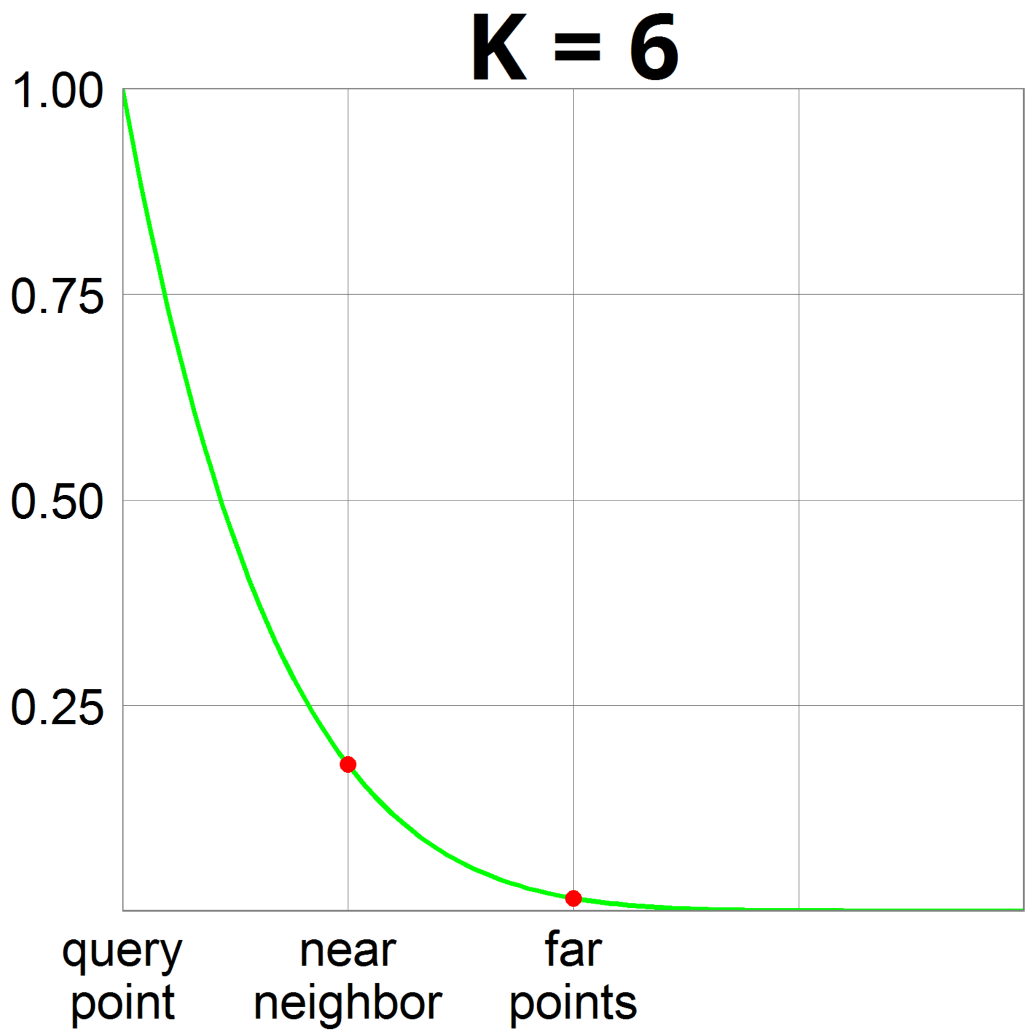

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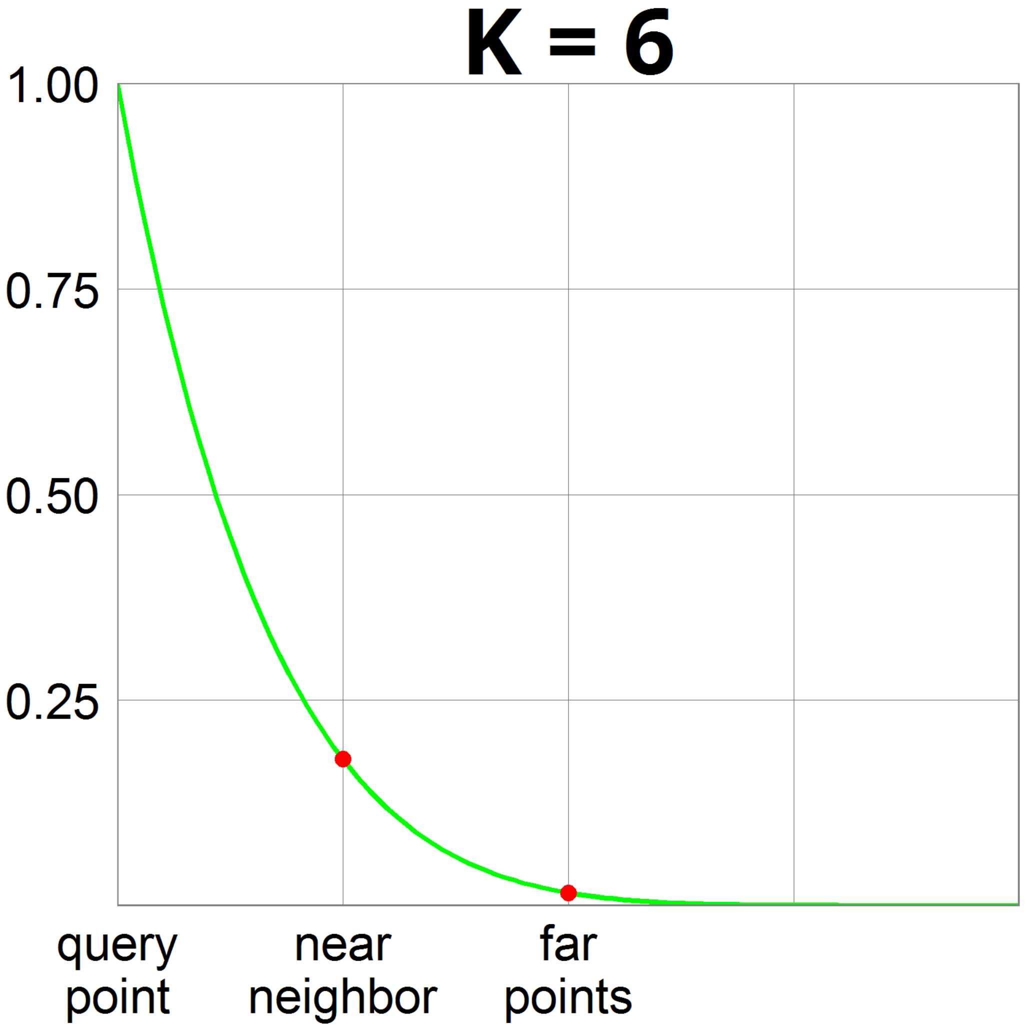
- Overall: **O($n^{1.42}$)** space, **O($n^{0.42}$)** query time, **K·L** hyperplanes



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Recap:

- $\mathbf{p_1}$ is collision probability for close pairs
- $\mathbf{p_2}$ — for far pairs

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Can we use this (significant) improvement in practice?

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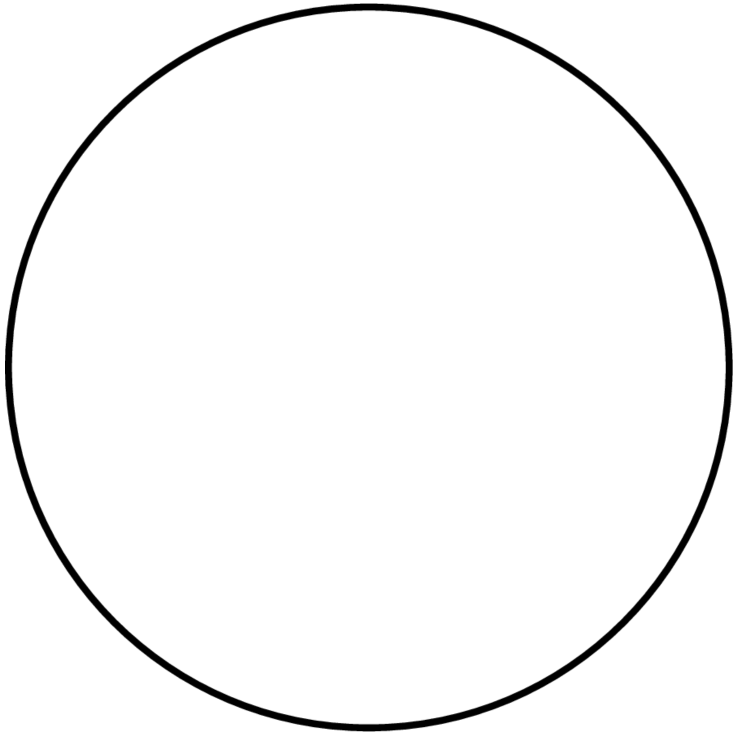

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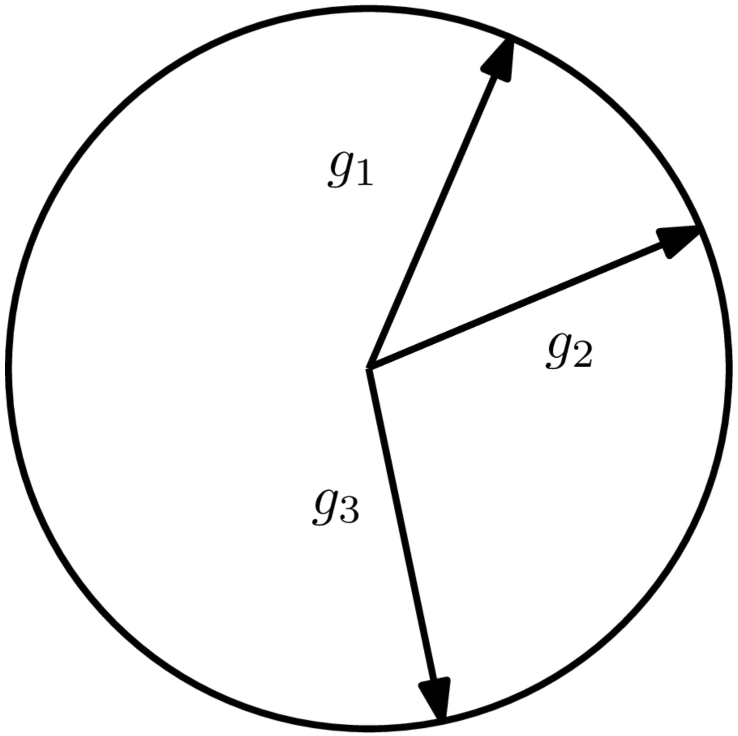

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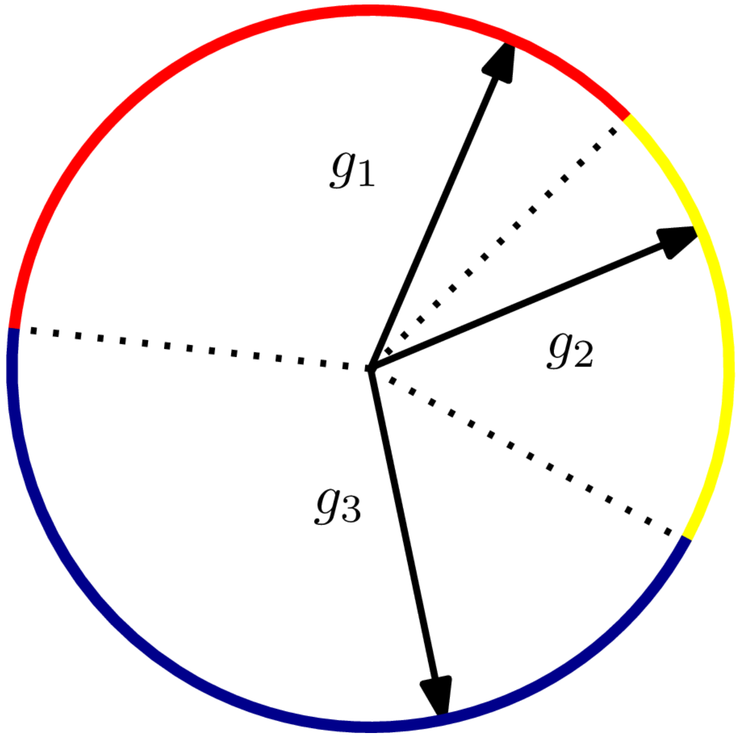

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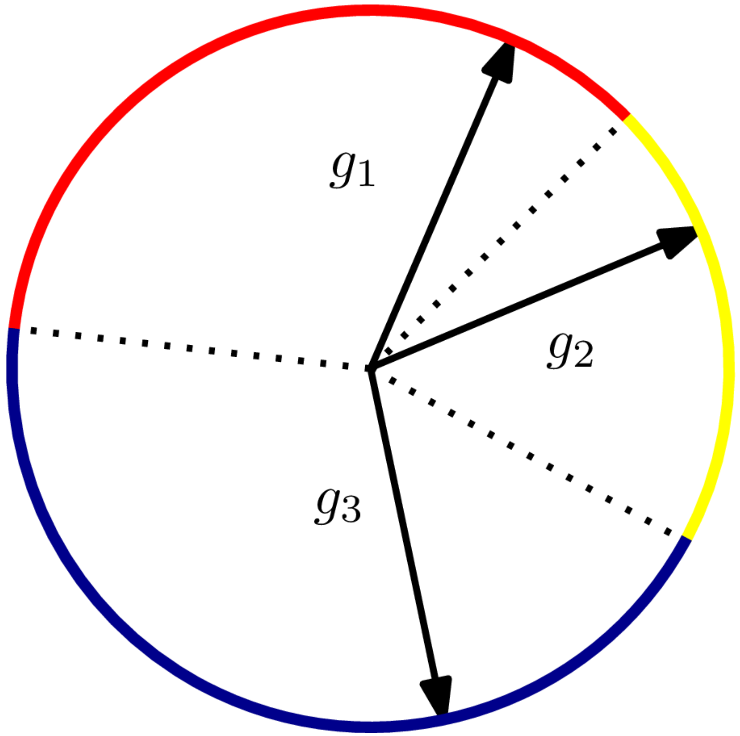
- **T = 2** is simply Hyperplane LSH



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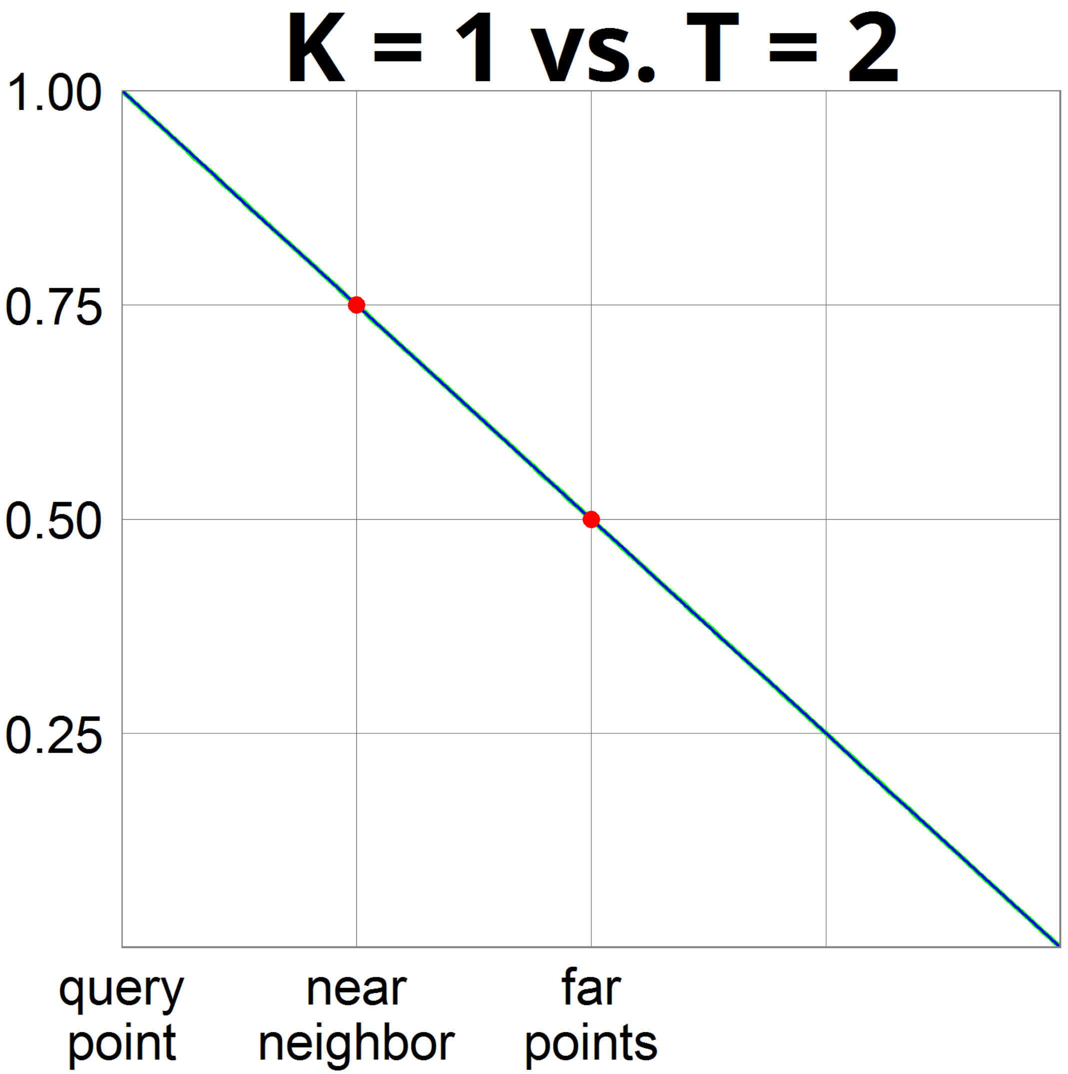

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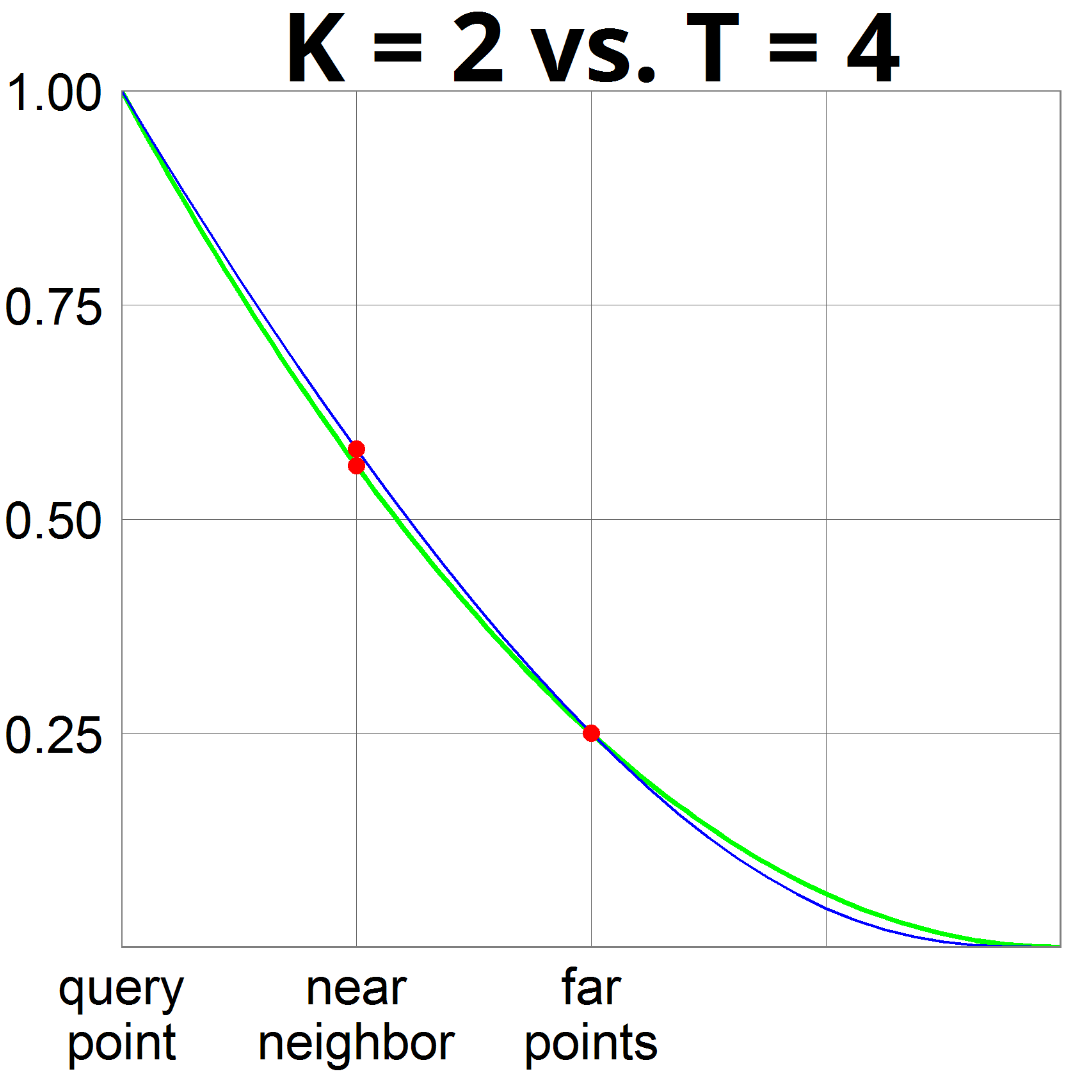

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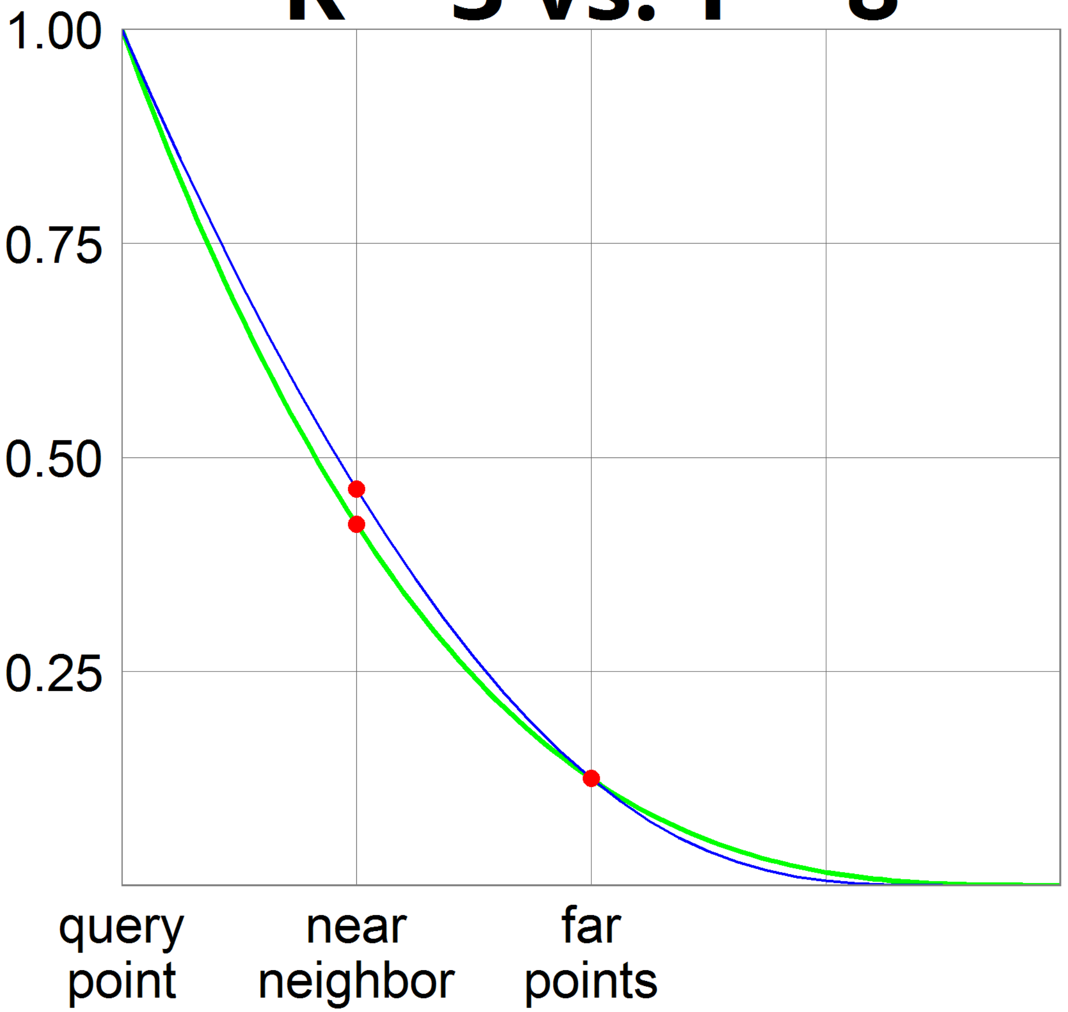

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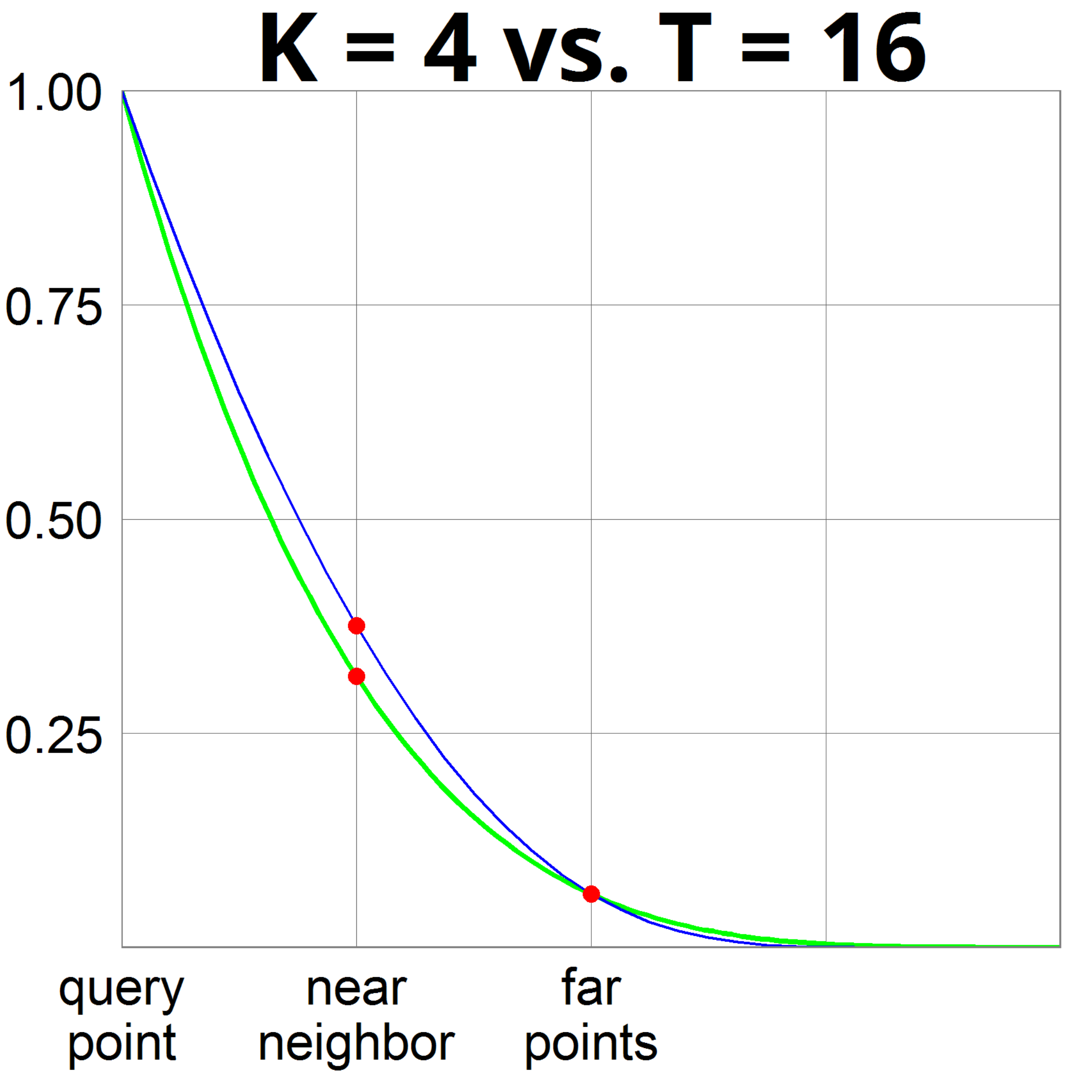

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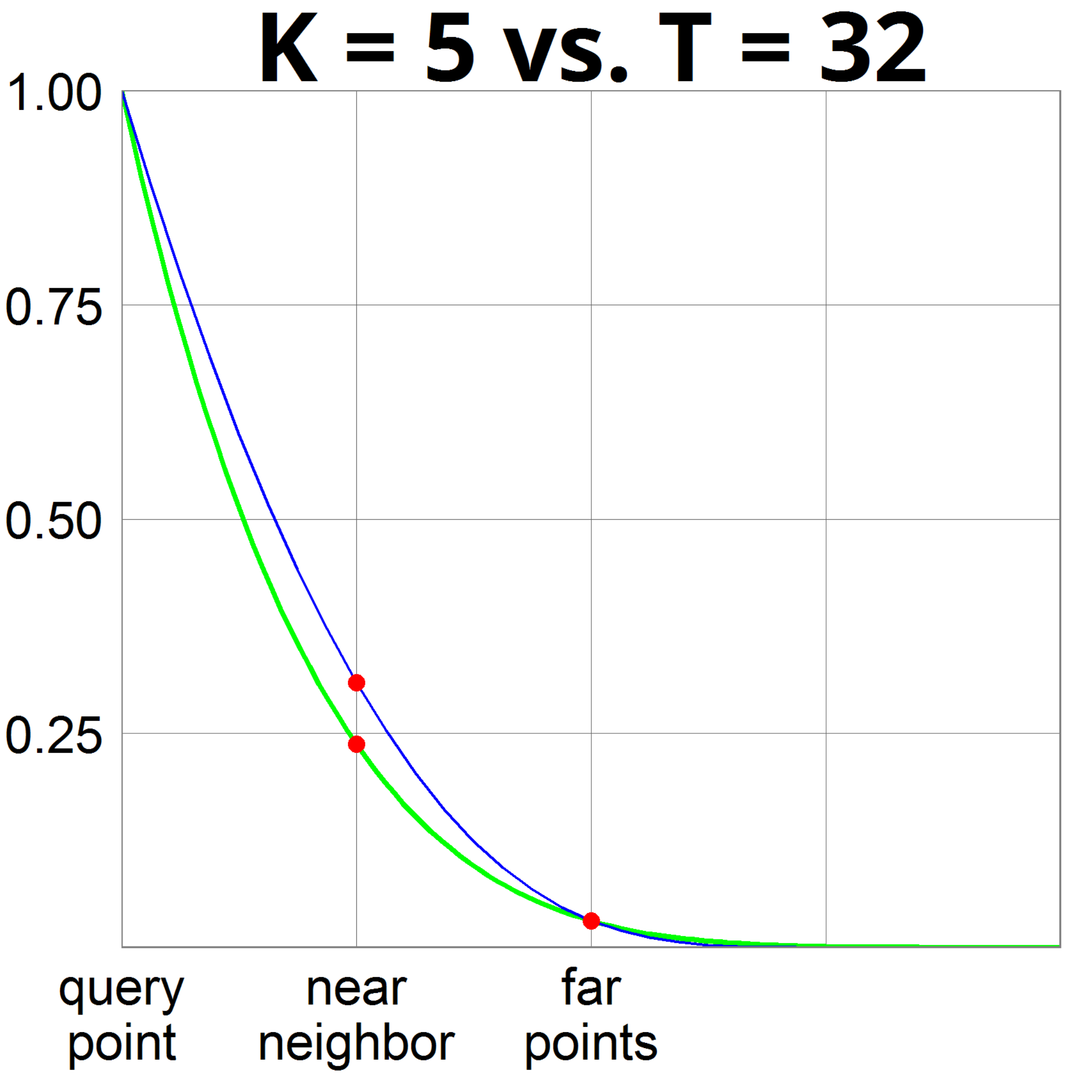

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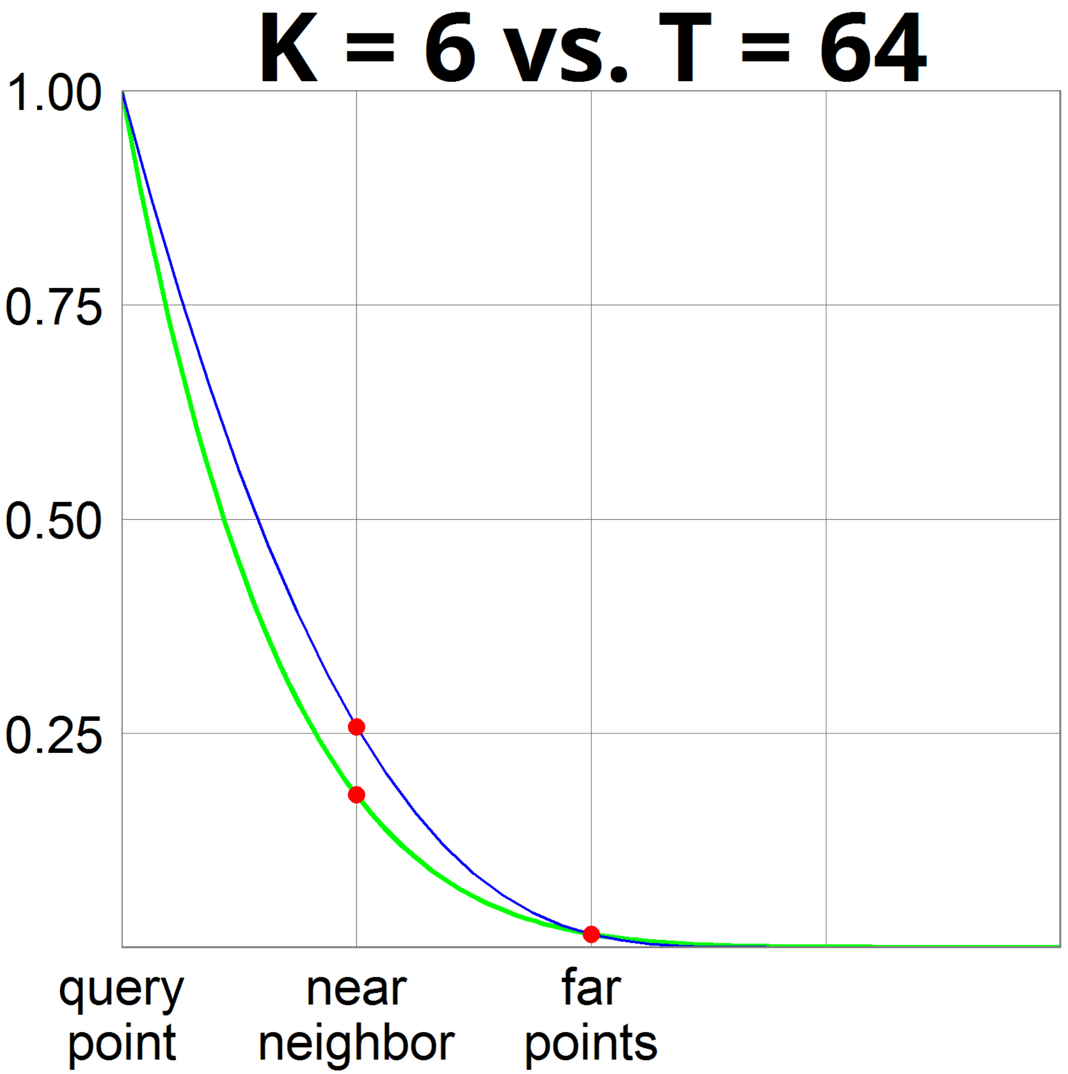

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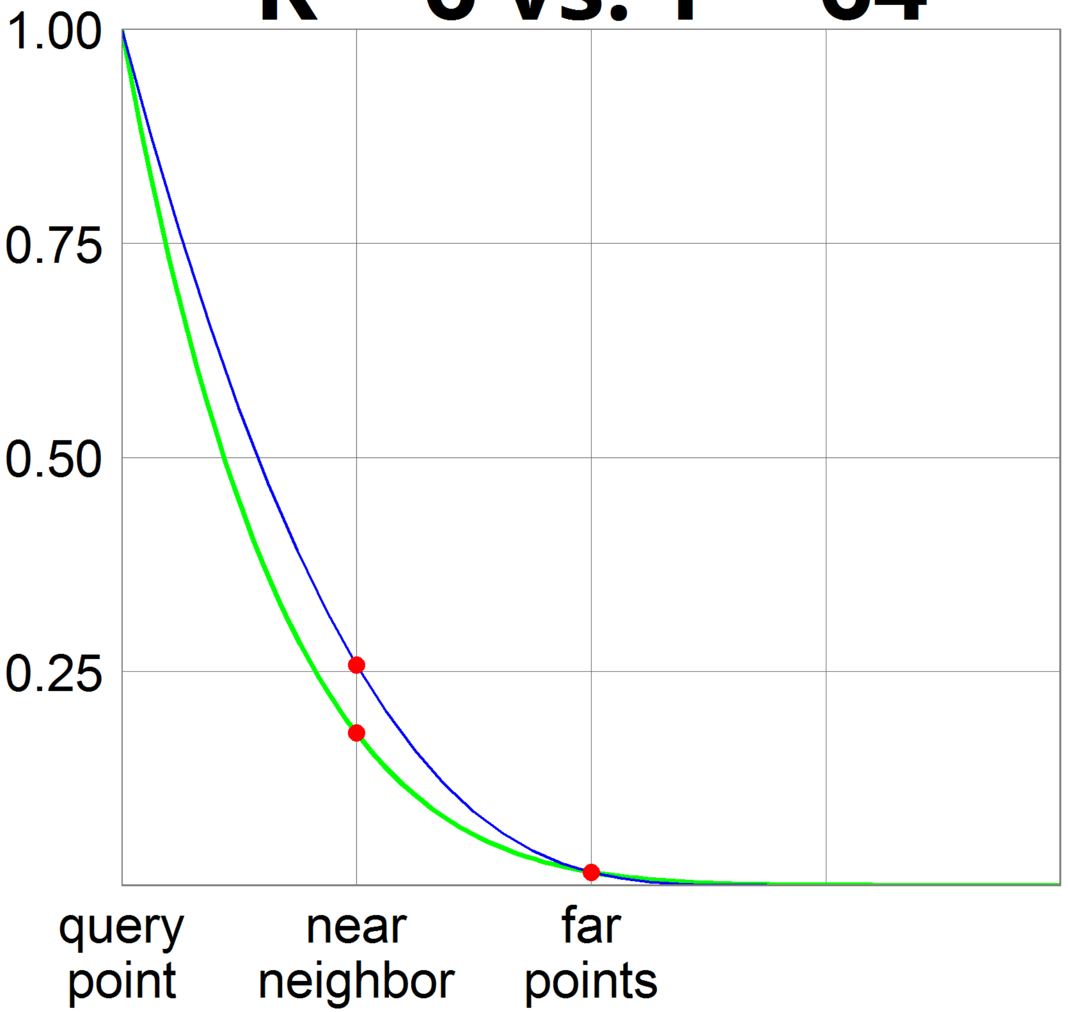
- Let us compare **K** hyperplanes vs. Voronoi LSH with **T = 2^K** (in both cases **K**-bit hashes)
- As **T** grows, the gap between Hyperplane LSH and Voronoi LSH increases and

$$\rho = \ln(1/p_1) / \ln(1/p_2)$$

approaches **0.18**

K = 6 vs. T = 64



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This work: yes!

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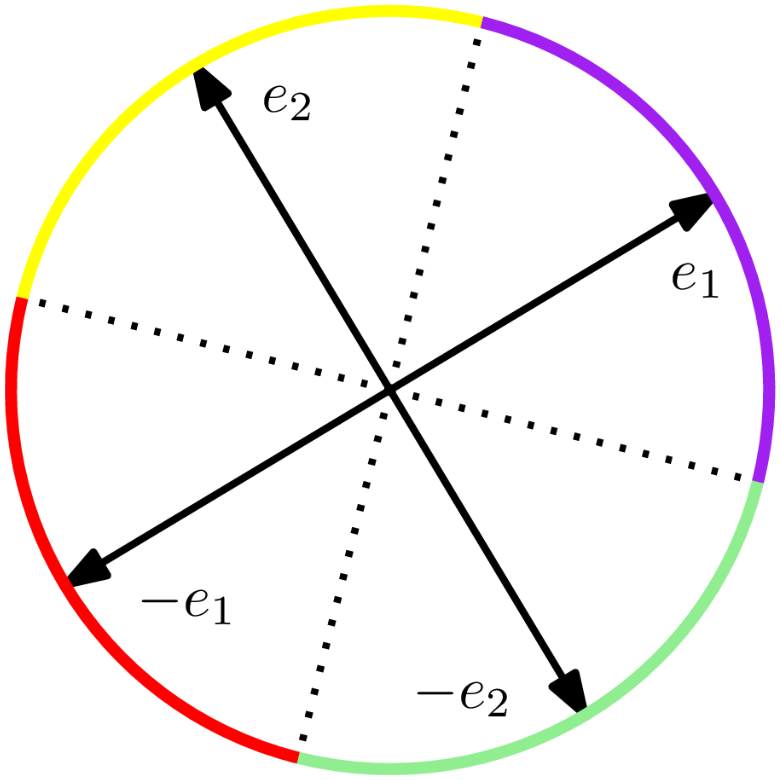

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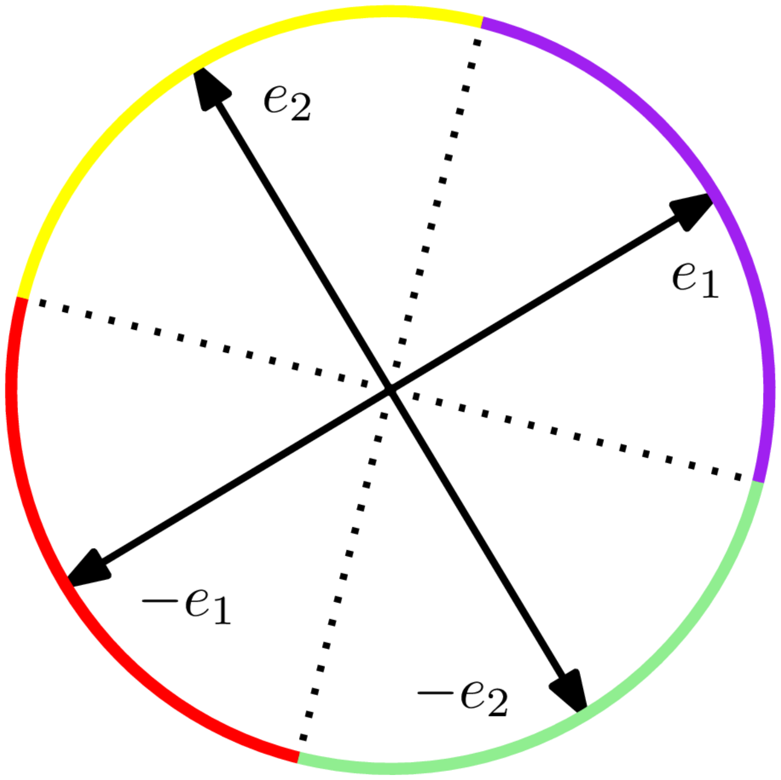

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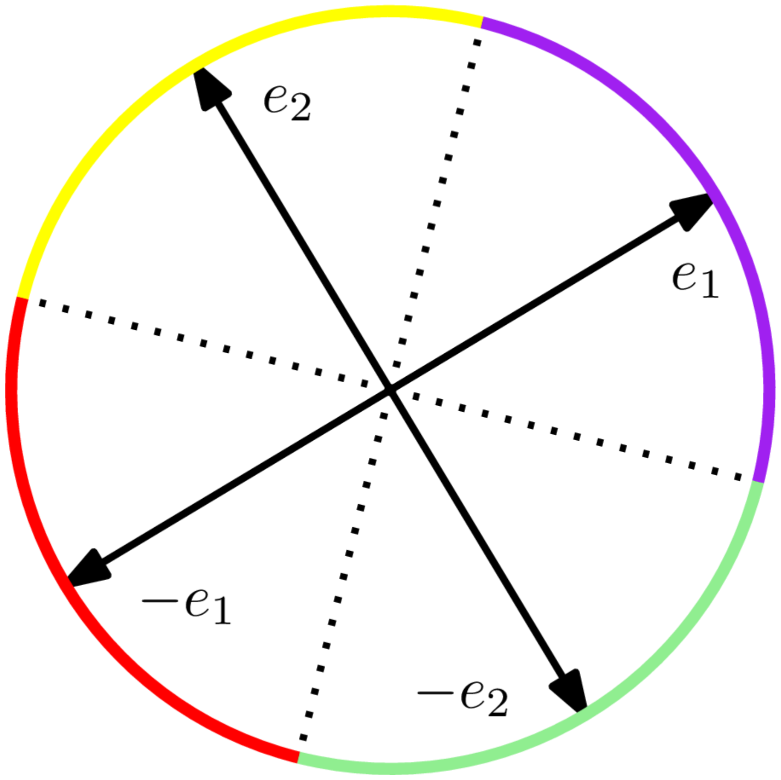

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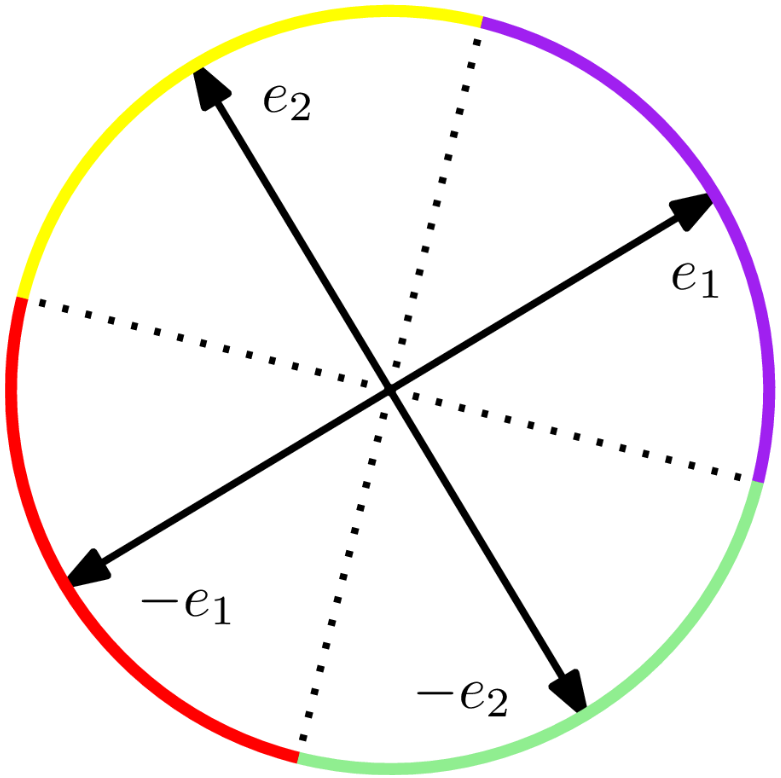

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Flip signs

$p' = (\pm p_1, \pm p_2, \ldots, \pm p_n)$

Hadamard

Hp'

Repeat (2-3 times)

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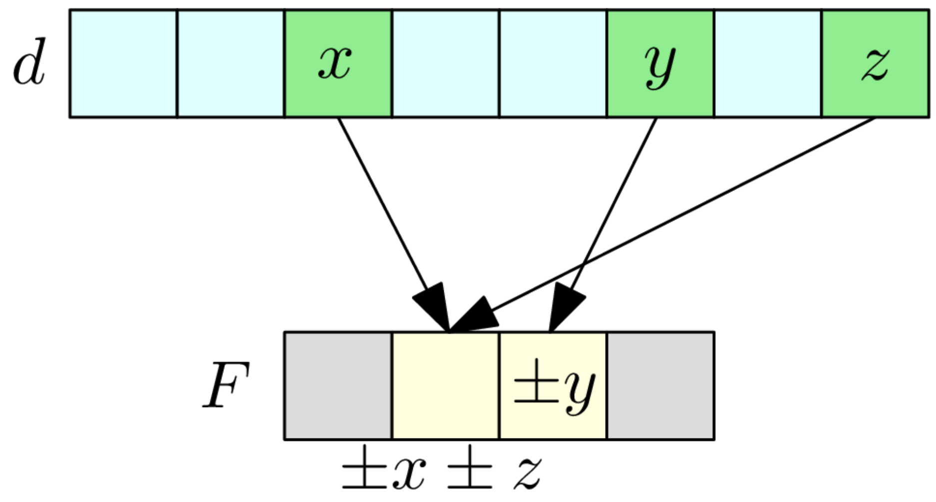

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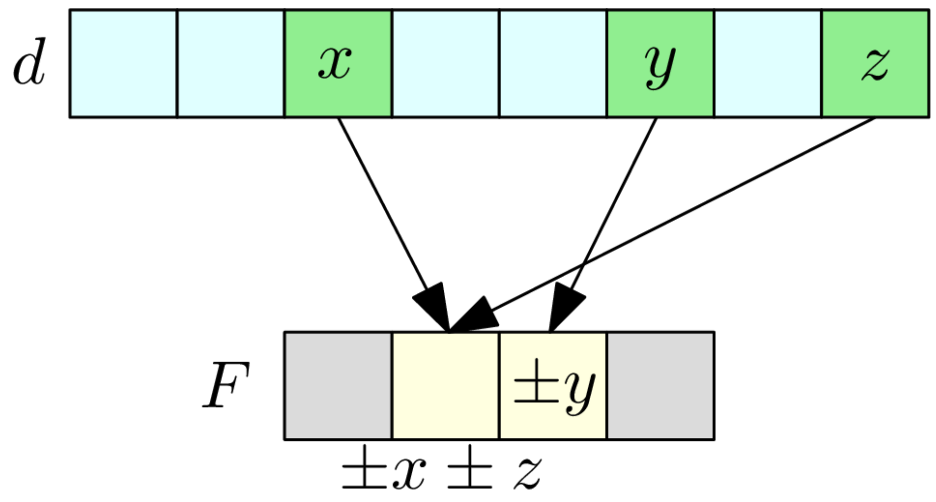

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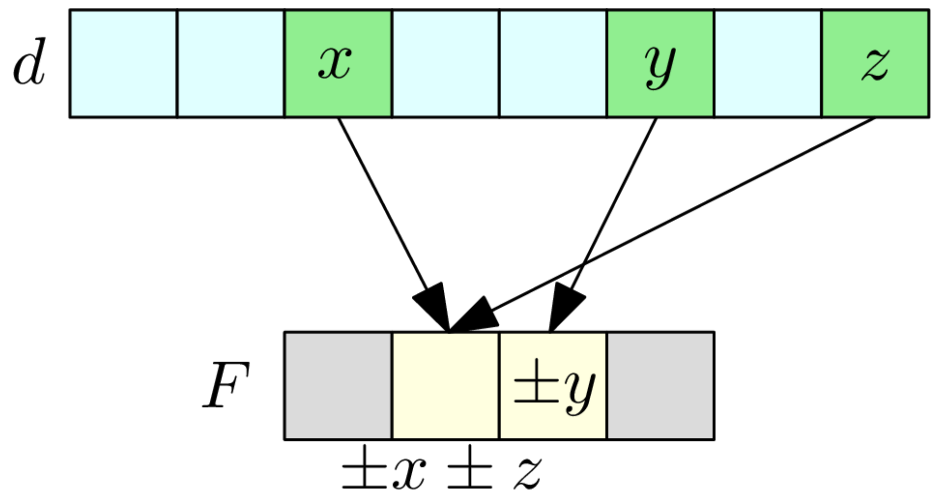

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- Available at **http://falconn-lib.org** together with Python bindings
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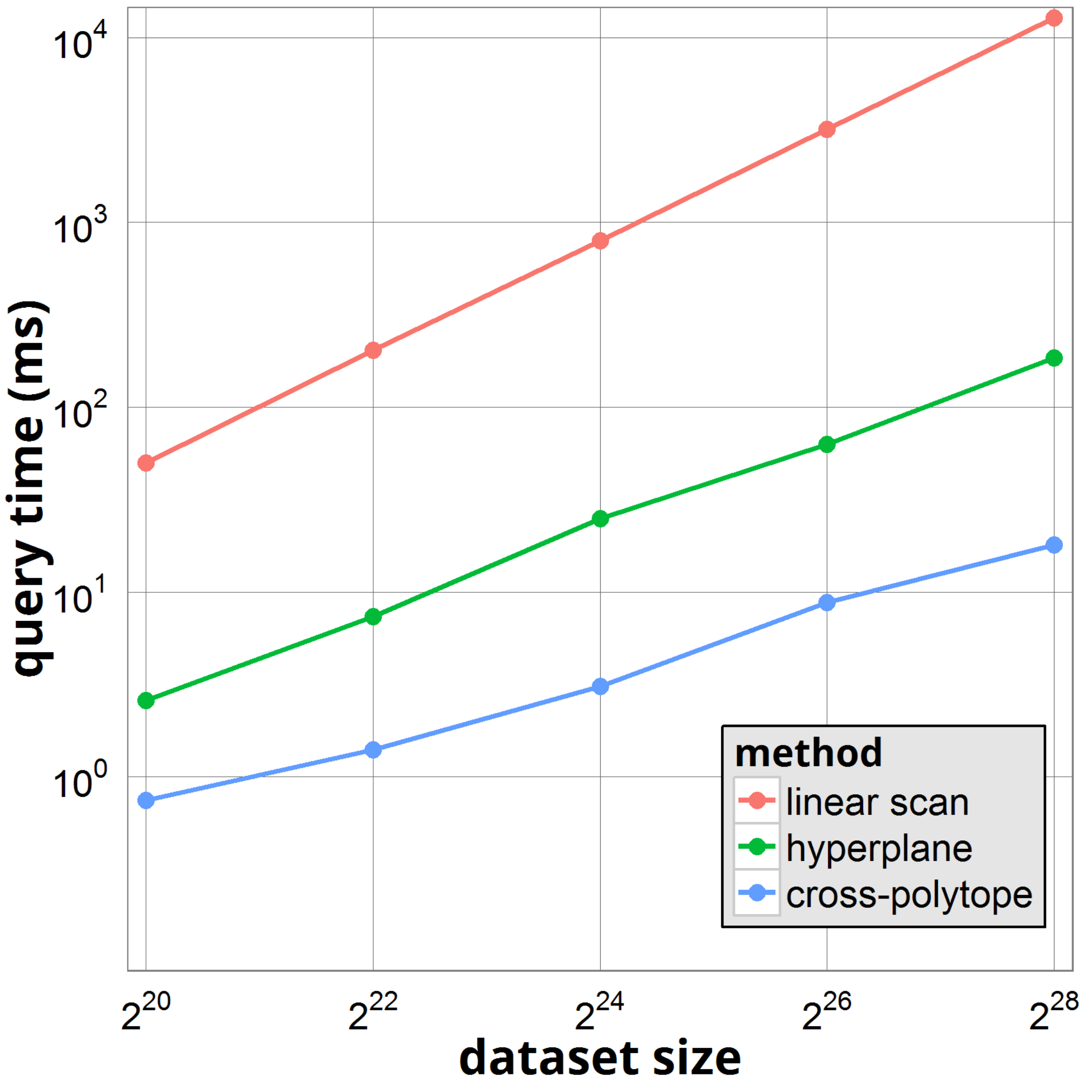

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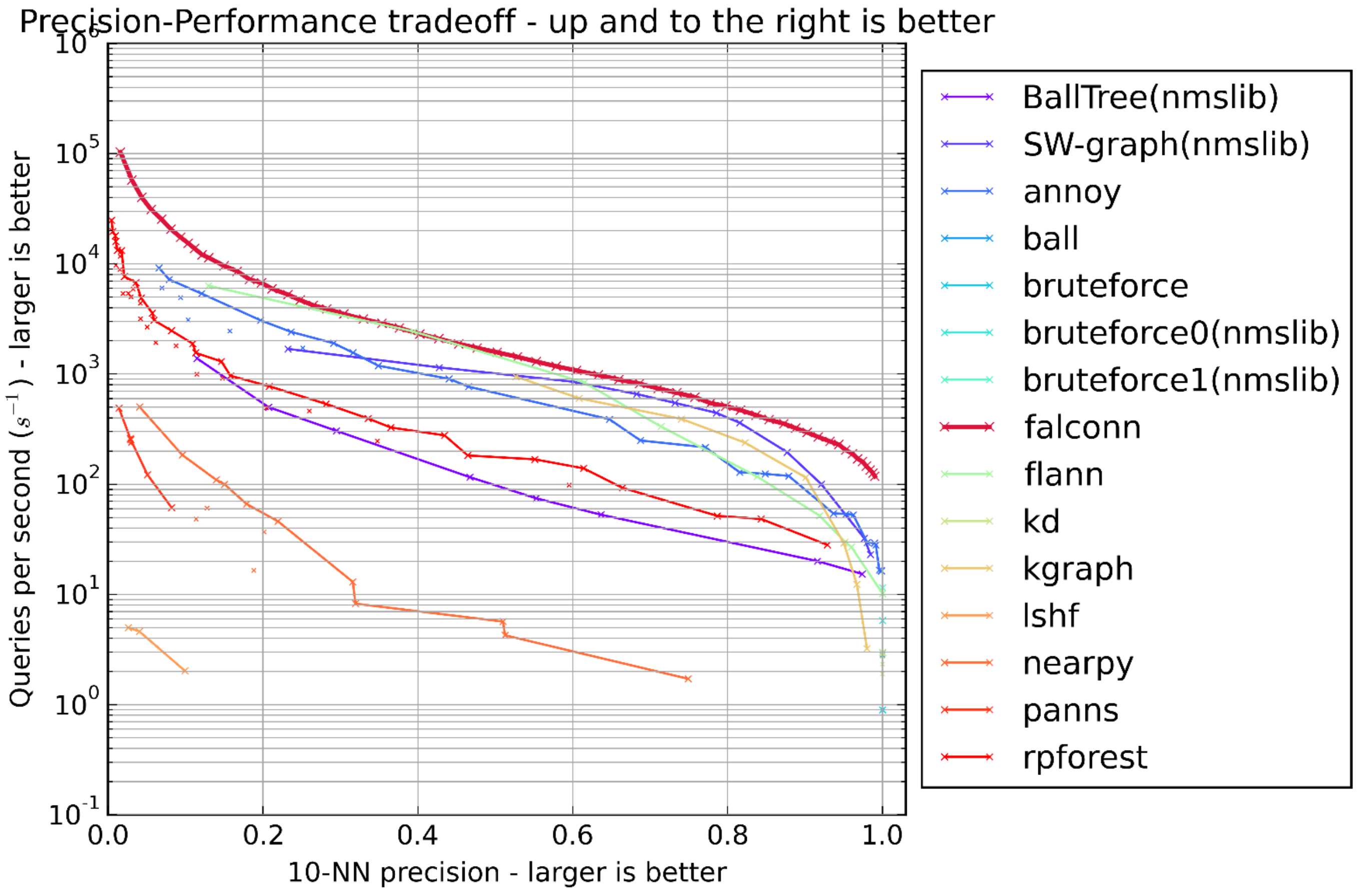

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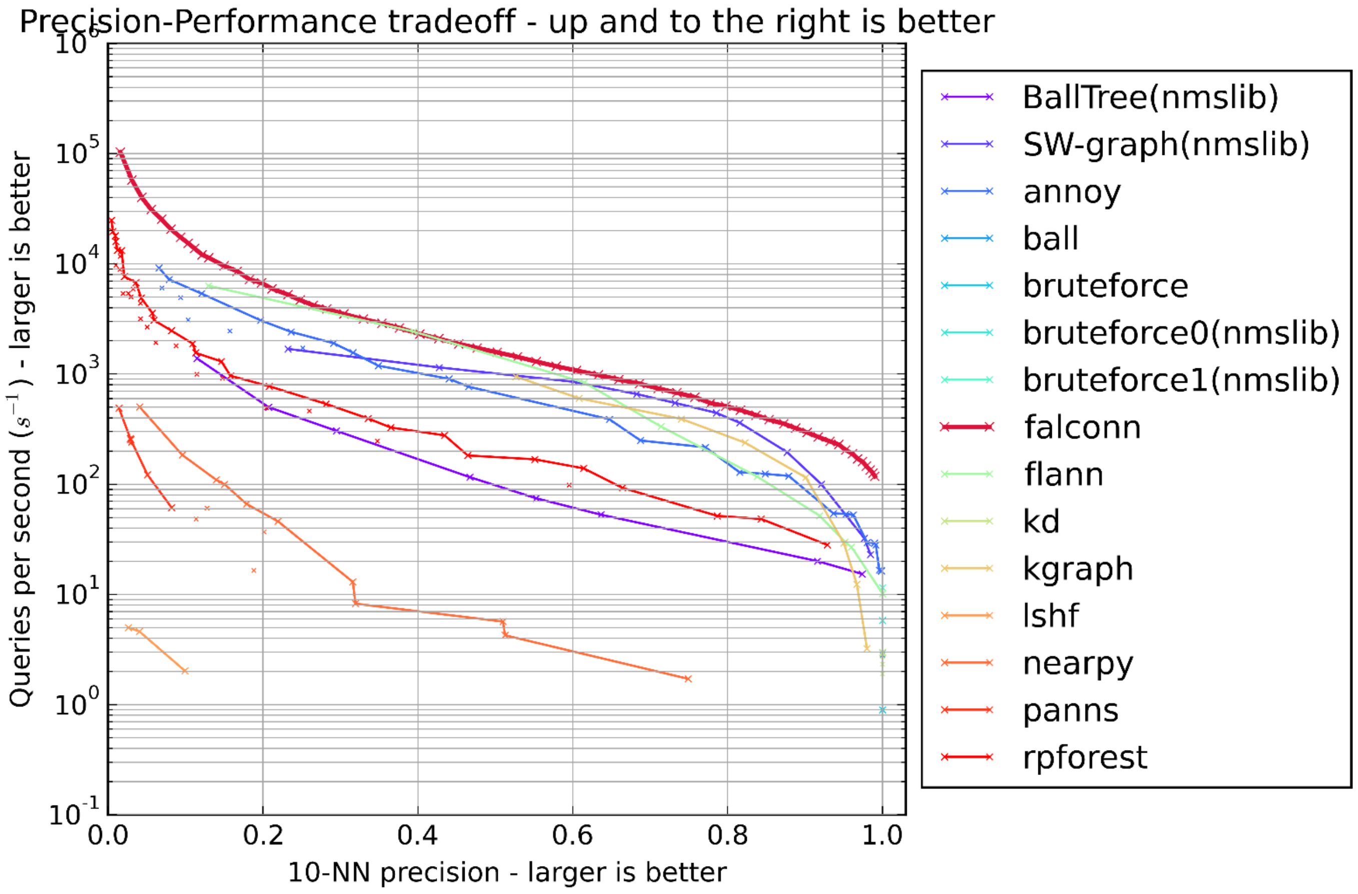

- **16**-bit hashes
- **1…1400** tables
- Single probe
- Accuracy **0.016…0.99**
- **10μs** to **8.5ms** query
- From **5 Mb** to **7 Gb**

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- For the further progress, need evaluation time sublinear in the range size!
  - Complexity of “decoding” for almost-orthogonal vectors

ANN with fast query time via sketches

Sketching metrics

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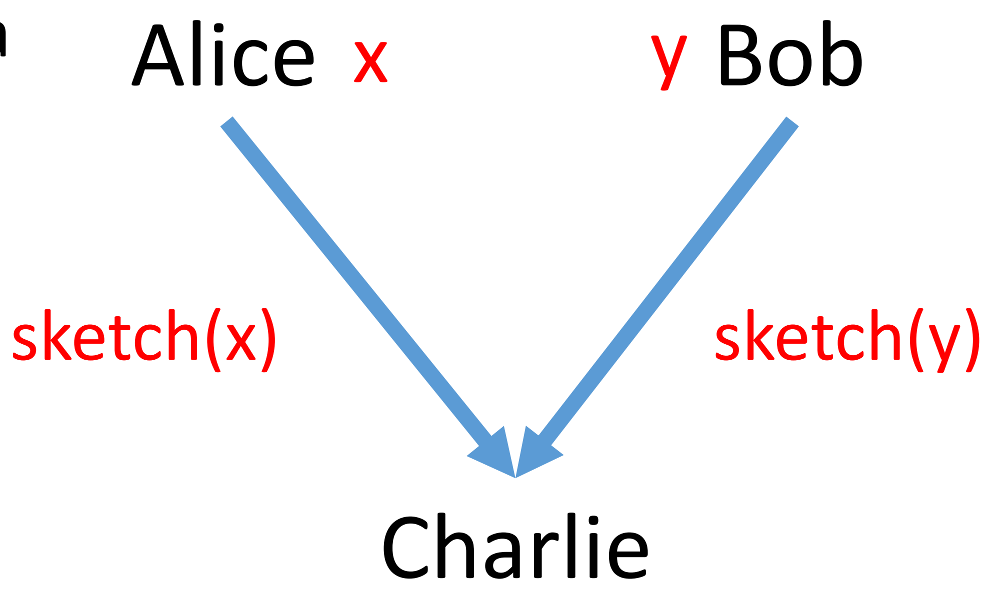

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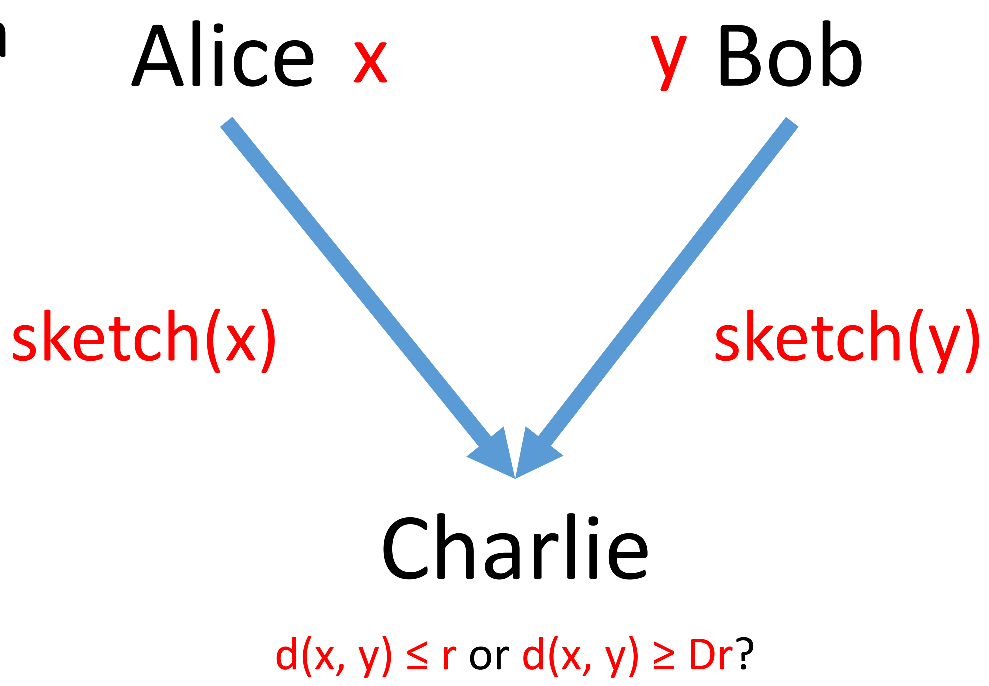

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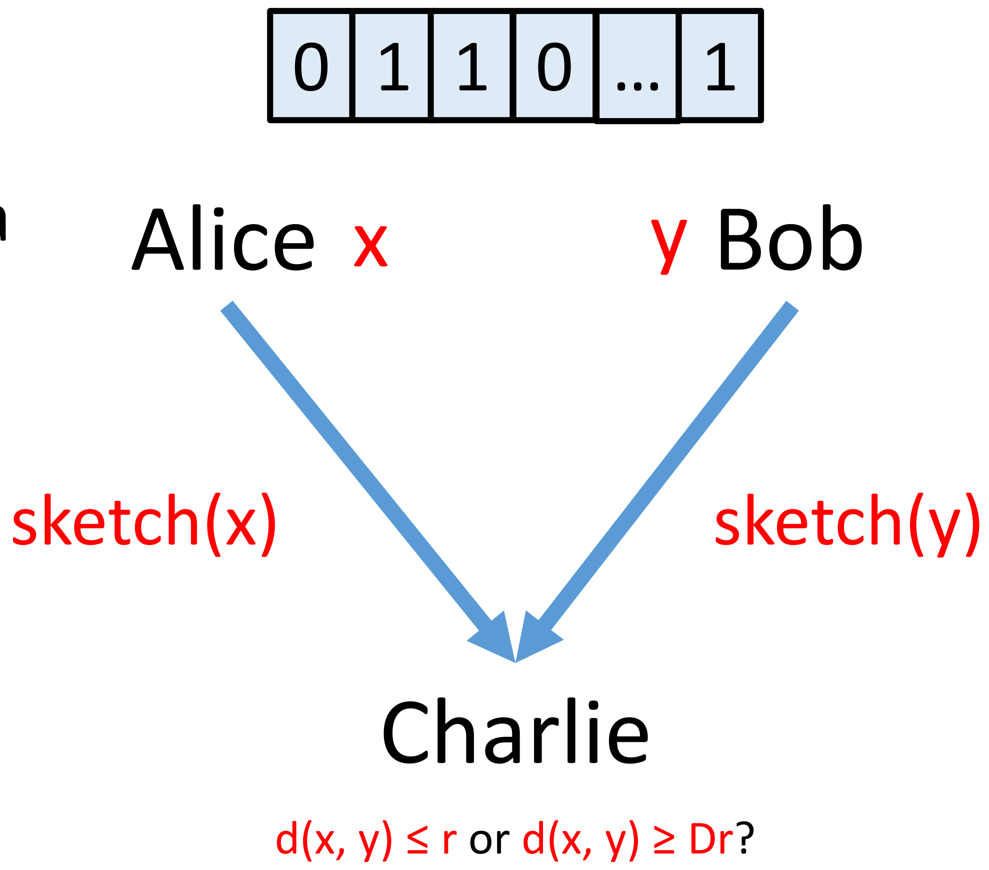

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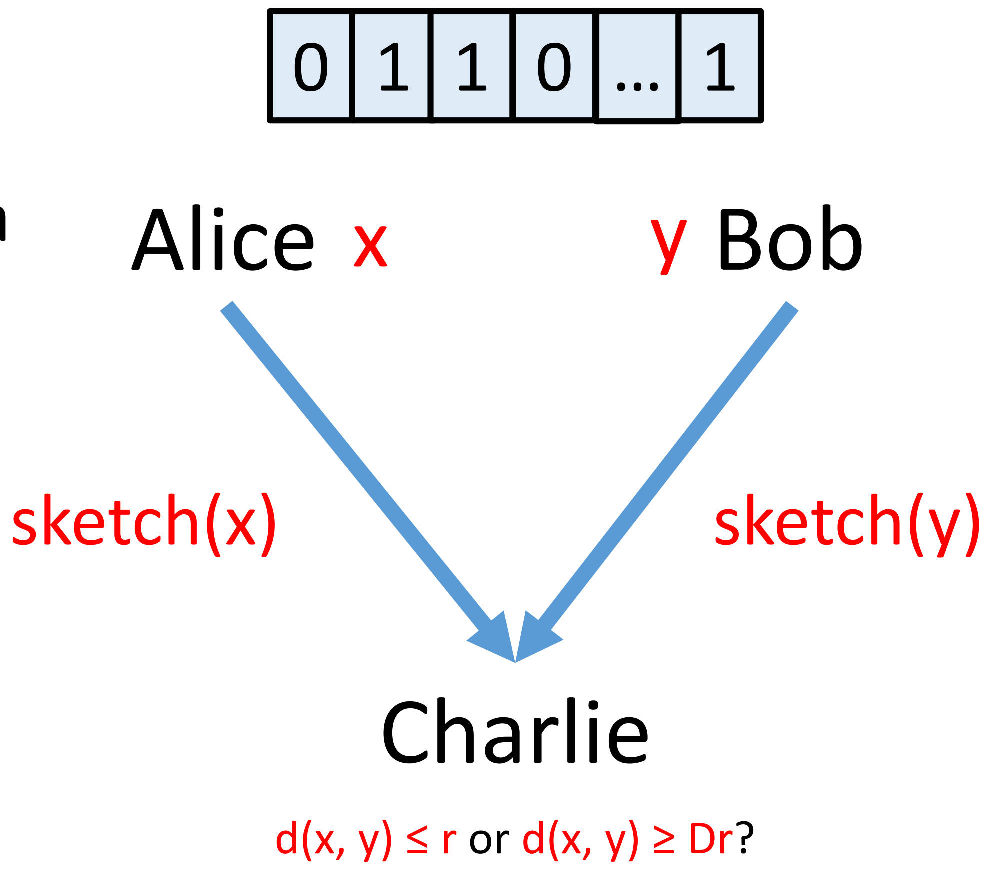

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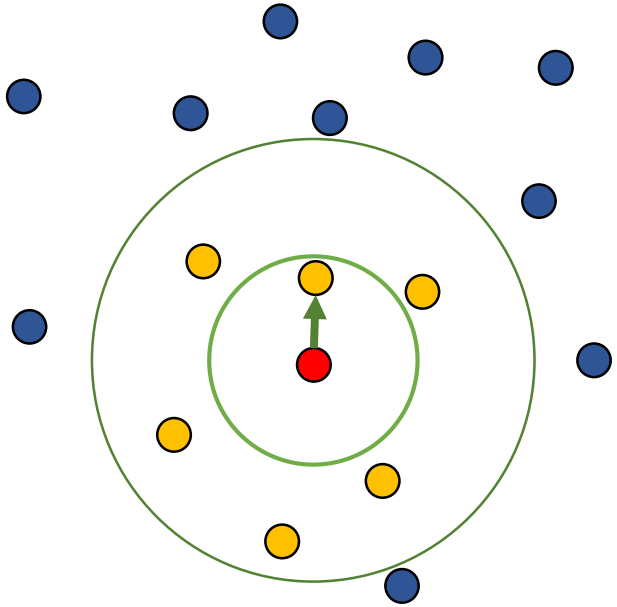

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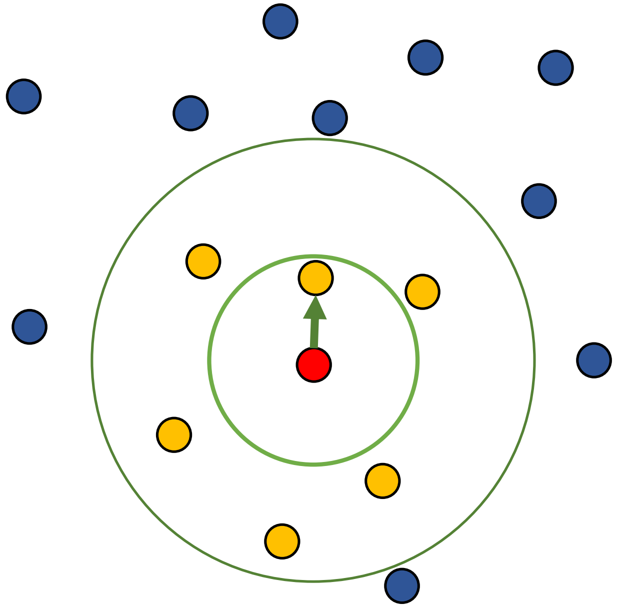

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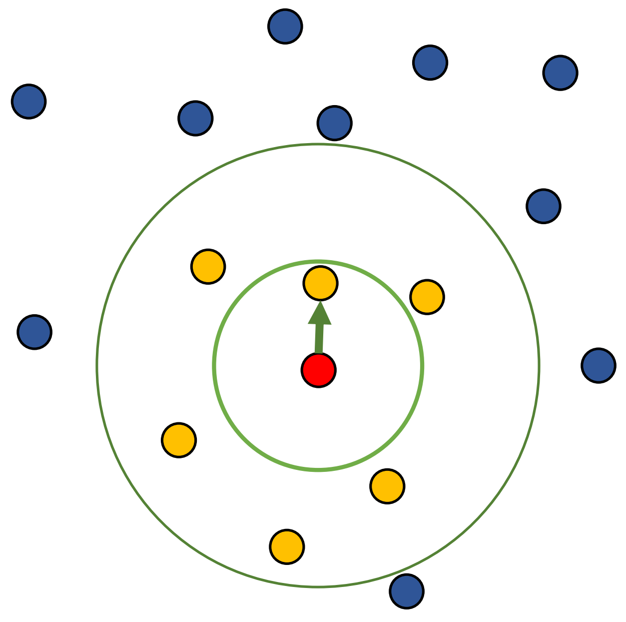

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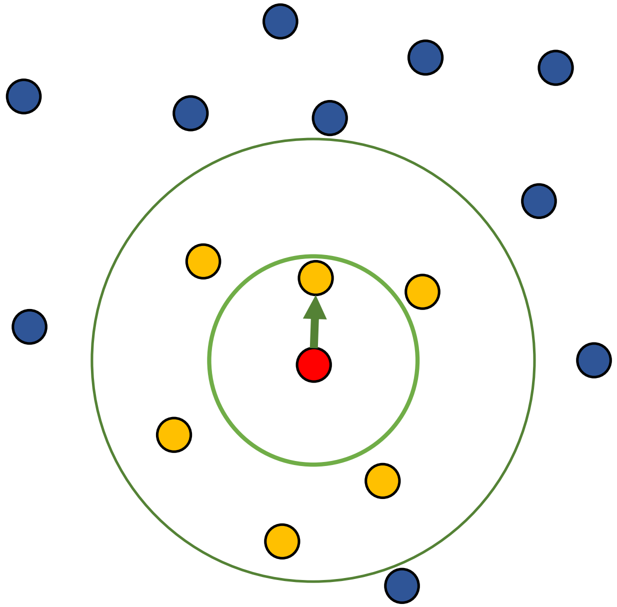
- For many metrics: the only approach



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- For $p > 2$ sketching ℓ_p is somewhat hard (Bar-Yossef, Jayram, Kumar, Sivakumar 2002), (Indyk, Woodruff 2005)
  - To achieve $D = O(1)$ one needs sketch size to be $s = \Theta\sim(d^{1-2/p})$

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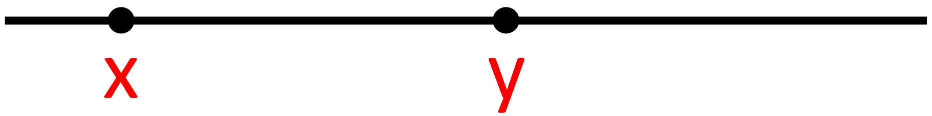

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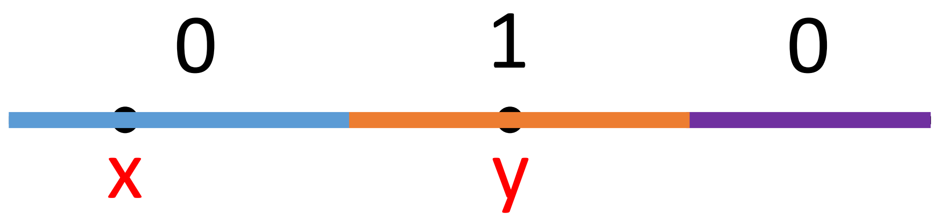

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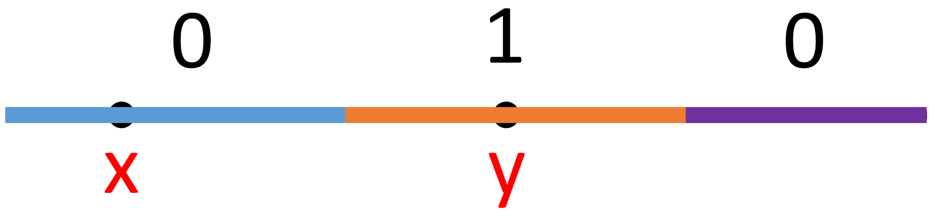

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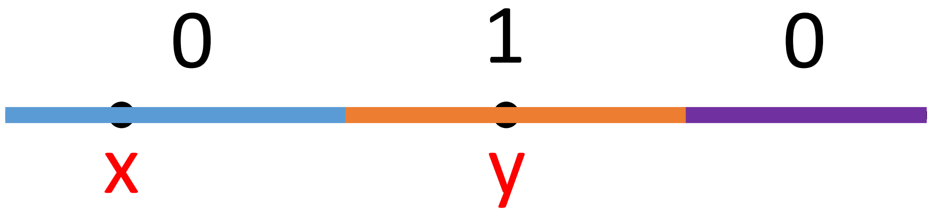

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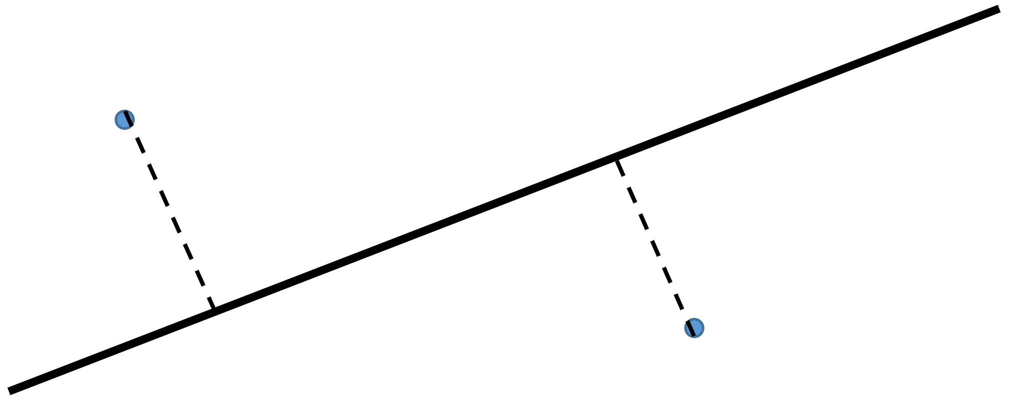

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- For $0 < p < 2$ use **p-stable distributions** instead
- Again, get $D = 1 + \varepsilon$ with $s = O(1 / \varepsilon^2)$

Summary

- Space: $\mathbf{n^{O(1/\varepsilon^2)}}$
- Query time: **poly(log n / ε)**