Streaming algorithms

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- **Example:** routers, possible statistics:
  - IP’s that are way more often that a typical one
  - etc.

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IP	Count
128.74.251.151	42
77.88.55.55	5
204.79.197.200	1

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  - Deterministic
  - Clearly optimal
- Drawbacks:
  - Two passes
  - Does not support deletions
  - Will see later a stronger guarantee (**L_2** vs **L_1**)

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- **Linear sketches:** **$sk(x) = Ax$** for random **A**
- Easy to maintain **Ax** under insertions/deletions

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- But let’s not worry about it!

Taxonomy of linear sketches

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- Graph sketches etc.

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  - For a hash **h: [u] → [t]**, store sums for each of the **t** bins

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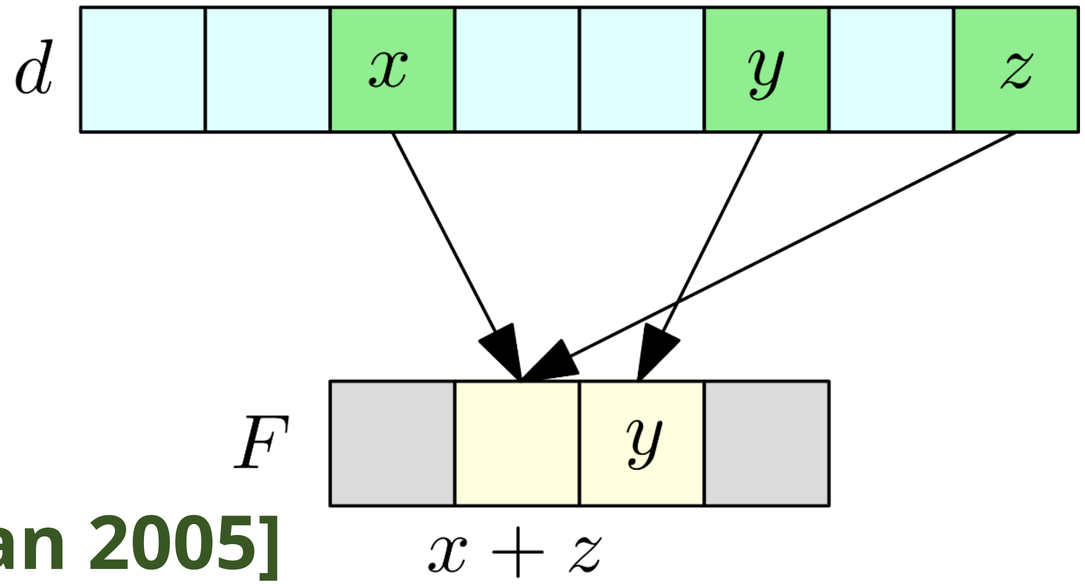

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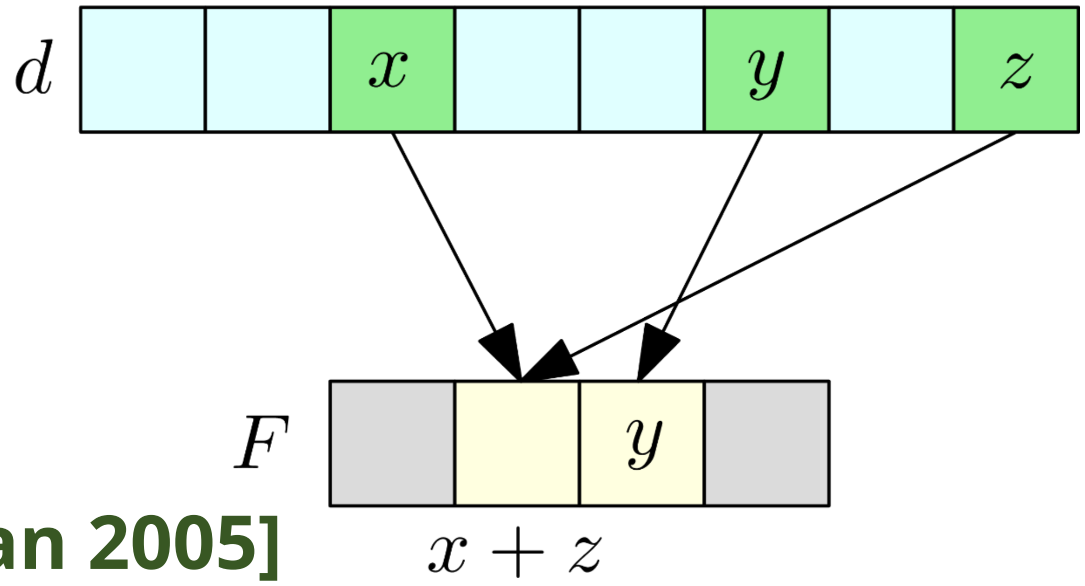

- **Idea: [Cormode, Muthukrishnan 2005]**
  - For a hash **h: [u] → [t]**, store sums for each of the **t** bins
  - Repeat for **s** hashes
  - For a point query **k**, take the **minimum** of the sums for **h(k)**
- **t=1/ε, k=O(log(1/δ))**

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  - Combine elements with random signs
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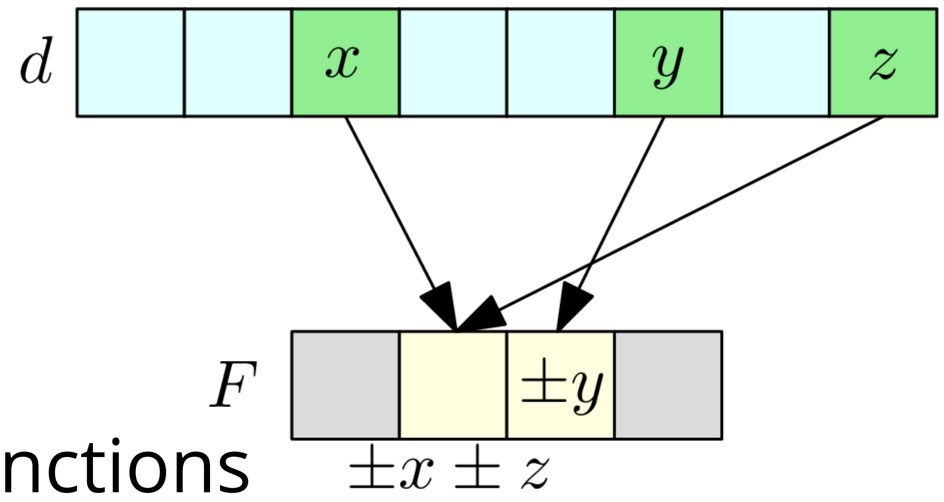

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- Finding connected components?

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- Even cooler: gives a way to compress (linearly) each row of an adjacency matrix into **polylog(n)** words
- The plan:
  - Design a “classical” algorithm that is amenable to sketching
  - Implement it using sketches

Classical algorithm

- Repeat **O(log n)** times:
  - For every node find an arbitrary neighbor
  - Collapse subgraphs formed by these pairs

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- **Remains: linear** sketch for finding a coordinate in the support of a vector
  - Linearity is crucial

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- Idea: **[Jowhari, Saglam, Tardos 2011]**
  - Subsample at all rates
  - Use **sparse recovery** to recover the (small) support of a subsample

Connectivity: a wrap-up

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- Need a fresh sketch at every iteration
    - The probability of failure is over the sketch construction

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- How can one even hope to prove a lower bound like this?

Communication complexity

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$x \in \{0,1\}^n$ $\qquad$ $y \in \{0,1\}^n$

$$f(x, y)$$

Disjointness

- $f(x, y) = \exists i : (x_i = y_i = 1)$
- Any *randomized* communication protocol requires $\Omega(n)$ communication
- A conceptual proof using **information complexity**

[Bar-Yossef, Jayram, Kumar, Sivakumar 2002]

Estimating max-norm

- Stream of elements from **{1, …, d}**, **maximum frequency**
- **[Alon, Matias, Szegedy 1995]: 1.99**-approximation requires $\Omega(d)$ space
- Alice and Bob want to solve **disjointness**
  - Alice performs her updates
  - Alice sends the memory to Bob
  - Bob performs his updates
  - Max-frequency **1** vs. **2**
- Even one-way lower bound for disjointness is enough **(easy!)**

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